Section 6.1

2.

$$P(G=g)=\sum_{t=0}^{4}P(G=g|T=t)P(T=t)$$

Given $T=t$, $G\sim Bin(t,1/2)$, so

$$P(G=g|T=t)=\binom{t}{g}(1/2)^t$$

where $g=0,1,\cdots,t$.

$$\begin{aligned}
P(G=0) &= 1*0.1+1/2*0.2+(1/2)^2*0.4+(1/2)^3*0.2+(1/2)^4*0.1=0.33125\\
P(G=1) &= 0+1/2*0.2+2*(1/2)^2*0.4+3*(1/2)^3*0.2+4*(1/2)^4*0.1=0.4\\
P(G=2) &= 0+0+(1/2)^2*0.4+3*(1/2)^3*0.2+6*(1/2)^4*0.1=0.2125\\
P(G=3) &= 0+0+0+(1/2)^3*0.2+4*(1/2)^4*0.1=0.05\\
P(G=4) &= 0+0+0+0+(1/2)^4*0.1=0.00625
\end{aligned}$$

G	0	1	2	3	4
$P(G=g)$	0.33125	0.4	0.2125	0.05	0.00625

4.

$$P(X=5|X+Y=12)=\frac{P(X=5,X+Y=12)}{P(X+Y=12)}=\frac{P(X=5,Y=7)}{P(X+Y=12)}=\frac{\binom{10}{5}\binom{10}{7}}{\binom{20}{12}}$$

5.

We have $X_1\sim Poisson(\lambda_1)$, $X_2\sim Poisson(\lambda_2)$, and $X_1+X_2\sim Poisson(\lambda_1+\lambda_2)$, so

$$\begin{aligned}
P(X_1=k|X_1+X_2=n) &= \frac{P(X_1=k)P(X_2=n-k)}{P(X_1+X_2=n)}\\
&= \frac{\frac{e^{-\lambda_1}\lambda_1^k}{k!}\frac{e^{-\lambda_2}\lambda_2^{n-k}}{(n-k)!}}{\frac{e^{-\lambda_1-\lambda_2}(\lambda_1+\lambda_2)^n}{n!}}\\
&= \binom{n}{k}(\frac{\lambda_1}{\lambda_1+\lambda_2})^k(\frac{\lambda_2}{\lambda_1+\lambda_2})^{n-k}
\end{aligned}$$

So given $X_1+X_2=n$, $X_1\sim Bin(n,\frac{\lambda_1}{\lambda_1+\lambda_2})$.

Section 6.2

1.

(a)

$$\begin{aligned} P(X=x) &= P(min(X_1,X_2)=x) \\ &= P(min(X_1,X_2)\geq x) - P(min(X_1,X_2)\geq x+1) \\ &= P(X_1\geq x)P(X_2\geq x) - P(X_1\geq x+1)P(X_2\geq x+1) \\ &= (\frac{7-x}{6})^2 - (\frac{6-x}{6})^2 \\ &= \frac{13-2x}{36} \end{aligned}$$

so by Bayes rule, we have

$$P(Y=y|X=x) = \frac{P(X=x,Y=y)}{P(X=x)} = \frac{\frac{1}{36}}{\frac{13-2x}{36}} = \frac{1}{13-2x}$$

if $x=y$, and

$$P(Y=y|X=x) = \frac{\frac{2}{36}}{\frac{13-2x}{36}} = \frac{2}{13-2x}$$

if $x<y$.

$$\begin{aligned} E(Y|X=x) &= \frac{x}{13-2x} + \sum_{y=x+1}^{6} y\frac{2}{13-2x} \\ &= \frac{1}{13-2x}[x+2\frac{(7+x)(6-x)}{2}] \\ &= \frac{42-x^2}{13-2x} \end{aligned}$$

(b)

$$\begin{aligned} P(Y=y) &= P(max(X_1,X_2)\leq y) - P(max(X_1,X_2)\leq y-1) \\ &= P(X_1\leq y)P(X_2\leq y) - P(X_1\leq y-1)P(X_2\leq y-1) \\ &= \frac{y^2}{36} - \frac{(y-1)^2}{36} \\ &= \frac{2y-1}{36} \end{aligned}$$

so by Bayes rule, we have

$$P(X=x|Y=y) = \frac{P(X=x,Y=y)}{P(Y=y)} = \frac{\frac{1}{36}}{\frac{2y-1}{36}} = \frac{1}{2y-1}$$

if $x=y$, and

$$P(X=x|Y=y) = \frac{\frac{2}{36}}{\frac{2y-1}{36}} = \frac{2}{2y-1}$$

if $x < y$.

$$\begin{aligned} E(X|Y=y) &= \frac{y}{2y-1} + \sum_{x=1}^{y-1} x\frac{2}{2y-1} \\ &= \frac{1}{2y-1}[y + 2\frac{y(y-1)}{2}] \\ &= \frac{y^2}{2y-1} \end{aligned}$$

4

(a)

$$EY = EE(Y|X) = E(\frac{1+X}{2}) = \frac{n+3}{4}$$

(b)

$$\begin{aligned} EY^2 &= E[E(Y^2|X)] \\ &= E[\frac{(x+1)(2x+1)}{6}] \\ &= \frac{1}{3}EX^2 + \frac{1}{2}EX + \frac{1}{6} \\ &= \frac{1}{3}\frac{(n+1)(2n+1)}{6} + \frac{1}{2}\frac{n+1}{2} + \frac{1}{6} \\ &= \frac{4n^2+15n+17}{36} \end{aligned}$$

(c)

$$VarY = EY^2 - (EY)^2 = \frac{7n^2+6n-13}{144}$$

so

$$SD(Y) = \frac{\sqrt{7n^2+6n-13}}{12}$$

(d)

$$P(X+Y=2) = P(X=1, Y=1) = \frac{1}{n}$$

6.

(a)

$$P(\forall X_i \le t|N=n) = \prod_{i=1}^{n} P(X_i \le t) = t^n$$

(b)

$$\begin{aligned} P(\forall X_i \le t) &= \sum_{n=0}^{\infty} P(\forall X_i \le t|N=n)P(N=n) \\ &= \sum_{n=0}^{\infty} t^n \frac{e^{-\mu}\mu^n}{n!} \\ &= e^{\mu(t-1)} \end{aligned}$$

(c)

$$P(S_N = 0) = P(N = 0) = e^{-\mu}$$

(d)

$$\begin{aligned} ES_N &= E[E(S_N|N = n)] \\ E(S_N|N = n) &= \sum_{i=1}^{n} EX_i = \frac{n}{2} \\ ES_N &= \frac{1}{2}EN = \frac{1}{2}\mu \end{aligned}$$

10.

Let X be the number of black balls, Y be the number of black balls drawn from another box. Then we have

$$X|Y = y \sim Bin(n, \frac{b+y}{b+w+d}) \qquad Y \sim Bin(d, \frac{b_0}{b_0+w_0})$$

so

$$\begin{aligned} EX = E[E(X|Y)] = E[\frac{b+Y}{b+w+d}n] &= \frac{bn}{b+w+d} + \frac{n}{b+w+d}EY \\ &= \frac{bn}{b+w+d} + \frac{n}{b+w+d}\frac{b_0}{b_0+w_0}d \end{aligned}$$

Section 6.3

1.

$$P(A) = \int P(A|X = x)f(x)dx = \int_0^1 x^2 dx = \frac{1}{3}$$

2.

(a)

$$f(x) = \int_0^1 f(x,y)dy = \int_0^1 (2x + 2y - 4xy)dy = 1$$

By symmetry, $f(y) = 1$.

(b)

$$f_y(y|X = \frac{1}{4}) = \frac{f(x,y)}{f_X(x)} = 2\frac{1}{4} + 2y - 4\frac{1}{4}y = \frac{1}{2} + y$$

(c)

$$\begin{aligned} E(Y|X = \frac{1}{4}) &= \int y f_y(y|X = \frac{1}{4})dy \\ &= \int_0^1 y(2x + 2y - 4xy)dy \\ &= \frac{7}{12} \end{aligned}$$

9.
(a)

$$\begin{aligned}
P(AB|Y=p) &= P(A|Y=p)P(B|Y=p) = p^2 \\
P(AB) &= \int P(AB|Y=p)P(Y\in dp) = \int_0^1 p^2 dp = \frac{1}{3} \\
P(B) &= \int P(B|Y=y)P(Y\in dp) = \int_0^1 p dp = \frac{1}{2} \\
P(A|B) &= \frac{P(AB)}{P(B)} = \frac{2}{3}
\end{aligned}$$

(b)

$$\begin{aligned}
P(Y\in dp|AB^c) &= \frac{P(Y\in dp, AB^c)}{P(AB^c)} \\
&= \frac{P(AB^c|Y\in dp)P(Y\in dp)}{\int P(AB^c|Y\in dp)P(Y\in dp)} \\
&= \frac{p(1-p)dp}{\int_0^1 p(1-p)dp} \\
&= 6p(1-p)dp
\end{aligned}$$