

Review Problems 2–Section 2.1.7 provided by Xu Han

Section 2.1.12-2.1.14 provided by Michael Freiman

Review Problems:

2. Let A denote the event that first ball drawn is white; Let B denote the event that at least one of the two balls drawn is white. By Bayes Rule, we have:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

Here $P(B|A) = 1, P(A) = \frac{1}{2}, P(A^c) = \frac{1}{2},$ and $P(B|A^c) = \frac{1}{3},$ because if first ball drawn is black, then we will add an additional black ball into the box, so the probability of the event that a second ball drawn is white will be $\frac{1}{3}.$ Hence $P(A|B) = \frac{3}{4}.$

5. For order (easy hard easy), we could either be successful in the first two tasks or fail in the first one but succeed in the next two. The probability of passing the test is: $P_1 = zh + (1 - z)hz.$

For order (hard easy hard), the argument is the same as above. The probability of passing the test is: $P_2 = hz + (1 - h)zh.$

$$\therefore P_1 - P_2 = hz(h - z) < 0, \therefore P_1 < P_2$$

We will choose order (hard easy hard) to maximize the probability of passing the test.

6. A and B are independent $\Leftrightarrow P(\overline{AB}) = \overline{P(A)P(B)}.$

$$\begin{aligned} P(A^cB) &= P(B) - P(AB) \\ &= P(B) - P(A)P(B) \\ &= P(B)(1 - P(A)) \\ &= P(A^c)P(B) \end{aligned}$$

$$\begin{aligned} P(AB^c) &= P(A) - P(AB) \\ &= P(A) - P(A)P(B) \\ &= P(A)(1 - P(B)) \\ &= P(A)P(B^c) \end{aligned}$$

$$\begin{aligned} P(A^cB^c) &= P(B^c) - P(AB^c) \\ &= P(B^c) - P(A)P(B^c) \\ &= P(A^c)P(B^c) \end{aligned}$$

so A^c and B, A and B^c, A^c and B^c are pairwise independent.

8.

(a) When exactly 5 deals are required, since it is without replacement, the probability is

$$P_1 = \frac{39}{52} * \frac{38}{51} * \frac{37}{50} * \frac{36}{49} * \frac{13}{48} \approx 0.0823$$

(b) Let B denote the event that 5 or fewer deals are required. B^c means that there are no hearts in the first 5 deals.

$$P(B^c) = \frac{39}{52} * \frac{38}{51} * \frac{37}{50} * \frac{36}{49} * \frac{35}{48} \approx 0.2215$$

so $P(B) = 1 - P(B^c) \approx 0.7785$.

(c) Let A_i denote the event that exactly i deals are required.

$$\begin{aligned} P(A_3|B) &= \frac{P(B|A_3)P(A_3)}{P(B)} \\ &= \frac{1 * \frac{39}{52} * \frac{38}{51} * \frac{13}{50}}{0.7785} \\ &\approx 0.1866 \end{aligned}$$

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(a)

Probability of the same type is $P_1 = (0.42)^2 + (0.1)^2 + (0.04)^2 + (0.44)^2 = 0.3816$

Probability of different types is $P_2 = 1 - 0.3816 = 0.6184$

(b)

$$P(1) = (0.42)^4 + (0.1)^4 + (0.04)^4 + (0.44)^4 = 0.0687$$

$$\begin{aligned} P(2) &= [(0.42)^2(0.1)^2 + (0.41)^2(0.04)^2 + (0.42)^2(0.44)^2 \\ &+ (0.1)^2(0.04)^2 + (0.1)^2(0.44)^2 + (0.04)^2(0.44)^2] * \binom{4}{2} \\ &+ [(0.42)^3(1 - 0.42) + (0.1)^3(1 - 0.1) + (0.04)^3(1 - 0.04) \\ &+ (0.44)^3(1 - 0.44)] * \binom{4}{1} \\ &\approx 0.5972 \end{aligned}$$

$$P(4) = P(A \cap B \cap AB \cap O) = (4!) * 0.42 * 0.1 * 0.04 * 0.44 \approx 0.0177$$

$$P(3) = 1 - P(1) - P(2) - P(4) \approx 0.3163$$

11. Let A denote the event that this is a fair coin, and B denote the event that the first time the coin lands heads, and the second time it lands tails. By Bayes Rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

where $P(A) = \frac{f}{n}$, $P(A^c) = \frac{b}{n}$, $P(B|A) = \frac{1}{2} \frac{1}{2} = \frac{1}{4}$, and $P(B|A^c) = \frac{2}{3} \frac{1}{3} = \frac{2}{9}$. So

$$P(A|B) = \frac{9f}{9f + 8b}$$

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7. For each time, let A denote the event that you win, B denote the event that I win, and C denote the event that we get the same number. By symmetry, $P(A) = P(B)$. It is easy to show that $P(C) = \frac{1}{6} * \frac{1}{6} * 6 = \frac{1}{6}$. From $P(A) + P(B) + P(C) = 1$, we could get that $P(A) = \frac{5}{12}$. The probability that you win at least four times is:

$$P = \binom{5}{4} \left(\frac{5}{12}\right)^4 \left(\frac{7}{12}\right) + \binom{5}{5} \left(\frac{5}{12}\right)^5 \approx 0.1005$$

12

(a) The probability of making exactly eight bets is the probability that the fifth red occurs on the eighth spin. This occurs when exactly four of the first seven bets are red, and then the eighth bet is also red. This probability is

$$\binom{7}{4} \left(\frac{18}{38}\right)^4 \left(\frac{20}{38}\right)^3 \left(\frac{18}{38}\right) \approx 0.122$$

(b) The probability of making more than eight bets is the probability that at most four of the first eight bets are red. This can be computed from the binomial distribution:

$$\begin{aligned} & \binom{8}{0} \left(\frac{20}{38}\right)^8 + \binom{8}{1} \left(\frac{18}{38}\right) \left(\frac{20}{38}\right)^7 + \binom{8}{2} \left(\frac{18}{38}\right)^2 \left(\frac{20}{38}\right)^6 \\ & + \binom{8}{3} \left(\frac{18}{38}\right)^3 \left(\frac{18}{38}\right)^5 + \binom{8}{4} \left(\frac{18}{38}\right)^4 \left(\frac{18}{38}\right)^4 \\ & \approx 0.693 \end{aligned}$$

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(a) Since both of the woman's alleles are B, the allele she passes along to the child must also be B. Similarly, both of the man's alleles are b, so he can only pass those alleles along to the child. So the child must be Bb, hence have brown eyes.

(b) The child gets a B allele or a b allele (equally likely) from the mother, and a b allele from the father. The probability of the child's having brown eyes is the probability of getting a Bb allele pair, since a BB allele pair is impossible. Since a Bb or bb pair is equally likely, the probability of the child's having brown eyes is 1/2.

(c) There are four allele combinations, each with possibility 1/4. They are BB, Bb, bB and bb. Since three of these lead to brown eyes, the probability is 3/4.

(d) Ignoring the child for the moment, $P(\text{woman is BB}) = 1/3$ and $P(\text{woman is Bb}) = 2/3$. Now let A be the event that the child has brown eyes. $P(A | \text{mother is BB}) = 1$, and $P(A | \text{mother is Bb}) = 1/2$. The probability of the child having brown eyes is $P(\text{woman BB})P(A | \text{woman BB}) + P(\text{woman Bb})P(A | \text{woman Bb}) = \frac{1}{3} * 1 + \frac{2}{3} * \frac{1}{2} = \frac{2}{3}$. The probability of the woman being Bb and the child having brown eyes is $\frac{2}{3} * \frac{1}{2} = \frac{1}{3}$. Dividing the two probabilities gives the conditional probability of $\frac{1/3}{2/3} = \frac{1}{2}$.

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(a) The offspring will be (Ts, Pw), hence will be tall and purple.

(b) When a Ts is crossed with a Ts, the possibilities are TT with probability 1/4, Tt with probability 1/2 and tt with probability 1/4. Similarly, when a Pw is crossed with a Pw, the possibilities are PP with probability 1/4, Pw with probability 1/2 and ww with probability 1/4. (Here, I am not making a distinction between Ts and tT or Pw and wP.) Since the height allele and the color allele are independent, there are nine possibilities:

Alleles	Probability	Appearance
TT, PP	1/16	tall, purple
TT,Pw	1/8	tall, purple
TT,ww	1/16	tall, white
Ts,PP	1/8	tall, purple
Ts,Pw	1/4	tall, purple
Ts,ww	1/8	tall, white
ss,PP	1/16	short, purple
ss,Pw	1/8	short, purple
ss,ww	1/16	short, white

Tallying everything up, we get 9/16 tall and purple; 3/16 tall and white; 3/16 short and purple; and 1/16 short and white.

(c) $P(\text{at least 2 plants tall and purple}) = 1 - P(\text{less than 2 plants tall and purple}) = 1 - (P(0 \text{ plants tall and purple}) + P(1 \text{ plant tall and purple}))$

$$1 - \left(\binom{10}{0} \left(\frac{7}{16}\right)^{10} + \binom{10}{1} \left(\frac{7}{16}\right)^9 \left(\frac{9}{16}\right) \right) \approx 0.996$$