Statistics 430 – Spring 2007
Solutions to first practice midterm

1a) Let A be the event that the cardholder pays in full, and let B_1, B_2, B_3 be the events that a cardholder owes less than \$100, \$100 to 500, or more than \$500, respectively. Then $P(A)=P(A|B_1)P(B_1)+P(A|B_2)P(B_2)+P(A|B_3)P(B_3)=(.5)(.3)+(.3)(.5)+(.1)(.2)=.15+.15+.02=.32$.

1b) This problem uses Bayes' Rule.

$$P(B_3|A)=\frac{P(A|B_3)P(B_3)}{P(A)}$$

$$=\frac{.02}{.32}=\frac{1}{16}$$

1c.i) The probability of the first person paying the bill in full and the second not doing so is (.32)(.68). This is also the probability of the second person paying in full and the first not doing so, so the overall probability is 2(.32)(.68)=.4352.

1c.ii) P(at least one owed >\$500

=P(first person owed >\$500)+P(second person owed >\$500)-P(both owed\ >\$500)

$$=\frac{1}{16}+\frac{1}{16}-\left(\frac{1}{16}\right)^2=\frac{31}{256}.$$

2) Denote heads by H and tails by T. There are seven possibilities, listed with their probabilities:

H, blue, blue: $\frac{1}{2}\times\frac{4}{6}\times\frac{3}{5}=\frac{1}{5}$
H, blue, green: $\frac{1}{2}\times\frac{4}{6}\times\frac{2}{5}=\frac{2}{15}$
H, green, blue: $\frac{1}{2}\times\frac{2}{6}\times\frac{4}{5}=\frac{2}{15}$
H, green, green: $\frac{1}{2}\times\frac{2}{6}\times\frac{1}{5}=\frac{1}{30}$
T, blue, blue: $\frac{1}{2}\times\frac{5}{6}\times\frac{4}{5}=\frac{1}{3}$
T, blue, green: $\frac{1}{2}\times\frac{5}{6}\times\frac{1}{5}=\frac{1}{12}$
T, green, blue: $\frac{1}{2}\times\frac{1}{6}\times\frac{5}{5}=\frac{1}{12}$

(You should be able to turn this into a tree diagram without too much trouble. I have not done so here.) Note that (T, green, green) is impossible, since there is only one green ball in Urn 2. The probability of at least one of the balls being blue is 29/30. The probability of the coin being heads and at least one ball being blue is 7/15. So the conditional probability is $\frac{7/15}{29/30}=\frac{14}{29}$.

3a) Note that N is at least 2 and no more than 4. (After 3 tosses, either the game is over or there are exactly two heads and one tails, so the next toss must end the game.) N=2 only if the first two tosses are tails, which has a probability of 1/4. N=4 if the first two tosses are two heads and one tails, which has a probability of 3/8. Since N must be 2, 3 or 4, $P(N=3)=1-\frac{1}{4}-\frac{3}{8}=\frac{3}{8}$.

N	2	3	4
Probability	1/4	3/8	3/8

3b) $P(N \geq 3) = P(N = 3) + P(N = 4) = \frac{3}{4}$. $P(N \geq 3$ and at least two heads$) = P(N=3$ and at least two heads$) + P(N=4$ and at least two heads$)$. If $N=4$, then the first three tosses must be two heads and a tails, so $P(N=4$ and at least two heads$) = P(N = 4) = \frac{3}{8}$. If $N=3$, that means either the first three tosses were heads (probability $\frac{1}{8}$) or the third toss was the second tails, which means it is not the case that at least two heads were observed. So $P(N=3$ and at least two heads$) = \frac{1}{8}$, whence $P(N \geq 3$ and at least two heads$) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$. Using the formula for conditional probabilities gives $\frac{1}{2} \Big/ \frac{3}{4} = \frac{2}{3}$.

Another way to solve the problem is to list all possible sequences of tosses and their probabilities. The sequences are HHH, HHTH, HHTT, HTHH, HTHT, HTT, THHH, THHT, THT and TT.

4a) There are $\binom{10}{5}$ possible ways to choose 5 of the 10 balls. To choose exactly three red balls, we have $\binom{5}{3}$ ways to choose the three red balls and $\binom{5}{2}$ ways to choose the remaining balls. So the probability of choosing exactly three red balls $\dfrac{\binom{5}{3}\binom{5}{2}}{\binom{10}{5}} = \dfrac{100}{252} = \dfrac{25}{63}$.

4b.i) Using the binomial distribution, the probability is $\binom{20}{10}(.5)^{10}(.5)^{10} = \binom{20}{10}(.5)^{20}$.

4b.ii) If at least 19 of the balls are red, then either 19 or 20 are red. So the conditional probability is $\dfrac{P(20)}{P(19)+P(20)} = \dfrac{\binom{20}{20}(.5)^{20}}{\binom{20}{19}(.5)^{20} + \binom{20}{20}(.5)^{20}} = \dfrac{\binom{20}{20}}{\binom{20}{19} + \binom{20}{20}} = \dfrac{1}{20+1} = \dfrac{1}{21}$.

4c) The probability of a green ball is .2. So the number of green balls in 400 draws is very close to a normal distribution with mean 80 and standard deviation $\sqrt{400 \times .2 \times .8} = 8$. The probability of at least 75 green balls is $1 - \Phi\left(\dfrac{75 - \frac{1}{2} - 80}{8}\right) = 1 - \Phi\left(-\dfrac{11}{16}\right) \approx .754$. (Without the continuity correction, the answer would be $1 - \Phi\left(\dfrac{75 - 80}{8}\right) = 1 - \Phi\left(-\dfrac{5}{8}\right) \approx .734$.)

5. If the insurance company sells 2,000 policies at $300 each, it has total revenues of $600,000. Hence it makes a $350,000 profit if it pays out at most $250,000 in claims, in other words, if no more than two of the insured die in the next year. Since n=2,000 is large and p=.002 is small, we use the Poisson approximation with μ=4. The probability of at most two claims is then

$$\frac{e^{-4}4^0}{0!}+\frac{e^{-4}4^1}{1!}+\frac{e^{-4}4^2}{2!}=e^{-4}(1+4+8)=13e^{-4}\approx .238.$$