How a False Probability Model Changed the World: Birth, Death, and Redemption of Black-Scholes

J. Michael Steele

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- Part III: When Models Shape Markets
  - (Enough Homework: This is more of a celebration, reflection, and — perhaps — a caution.)

Part I: Beginning with an Almost Impossible Question

Consider a world where there is a stock and a “contingent claim”.

The stock costs 2 dollars at time zero, and at time 1 it is worth

- either 4 dollars (if it goes up)
- or 1 dollar (if it goes down)

The claim is costs X dollars at time zero, and at time 1 it is worth

- either 3 dollars (if the stock goes up)
- or 0 dollars (if the stock goes down)

Question: What is X?

Some Reasoning about the Almost Impossible Question

- Sure! Let P_{UP} denote the probability that the stock goes up. In that case, a pretty reasonable price for the contingent claim would be

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- Bad News: *Nobody* knows P_{UP}. It looks like we are stuck, and we all should soak for a moment in a bath of hopeless despair!

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- This turns out to be a marvelously fecund idea.

Solving For X

Table: Replication of a Derivative Security

	Portfolio	Derivative Security
Original cost	$\alpha S + \beta B$	X
Payout if stock goes up	$4\alpha + \beta$	3
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- Bottom Line: The unique arbitrage-free price for the contingent claim X is one dollar.

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- SUPER BONUS. This extremely simple example carries through to the real world.

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The gambler then guesses

$$X = P'_{\mathrm{UP}} * 3 + (1 - P'_{\mathrm{UP}}) * 0 = 1$$ and his guess is RIGHT!

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  - The THEOREM is about entirely arbitrage.

Introducing Black–Scholes World

The theory of option pricing owes a fundamental debt to Fisher Black and Myron Scholes who in 1973 considered the model P that now many people call "Black–Scholes World":

$$dS_t = \mu S_t \, dt + \sigma S_t dB_t \quad \text{and} \quad d\beta_t = r\beta_t \, dt$$

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- Our world has a finite horizon, T. Thus, $\tau = T - t$ is the "time left".

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- This almost defies credibility — yet still holds water.

The Famous Formula as a Special Case

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- If H is any function of the stock-price path S_t, $0 \leq t \leq T$, we can consider a contingent claim that pays us that function of the path. For example, the maximum, or the last value, or $\max(S_T - K, 0)$ in the case of the European call option.

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- For the European call option, the payout function just depends on the value of the stock at the terminal time.
- Given the stock price at time t, the conditional distribution of S_T given $S_t = S$ is just a log normal, so we can easily work out the expectation (1).

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- This is just the (interest rate adjusted) value of $E_{P'}(S_{[0:T]})$ in its concrete form; the famous Black-Scholes formula for a European call option.

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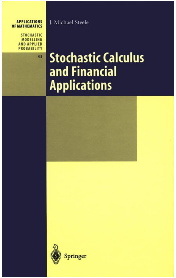
- Details have been omitted — but no crucial ideas

Brief Word from Our Sponsor

There are now many good places to learn stochastic calculus and its applications to mathematical finance, but ....

There is one we most warmly recommend:

- Friendly and honest
- Rigor without tedium
- Fun for the whole family



Sure, you could get other books, but don't you deserve the best?

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- More likely, the key driver was the explicit recipe for hedging. This is honest and operational, even absent a "formula."
- What else? Emergence of "volatility" as a central concept — perhaps THE central concept — in financial modeling.

Volatility — and Implied Volatility

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$$dS_t = \mu S_t\,dt + \sigma S_t dB_t \quad \text{and} \quad d\beta_t = r\beta_t\,dt$$

- One might be tempted to estimate σ from the sample path of the stock price and then plug in to get the option value.
- Gad Zooks — This does not work. Are we hosed?
- No. We reverse the process and get the "Implied Volatility"
- Strange? Yes. Useful? Yes. Universal? Absolutely.

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- Note (reluctantly) that empirical volatility and implied volatility are imperfectly related.
- Worry a little ... while a 273 trillion dollar market evolves in less than 30 years.

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- What is the empirical story for asset returns?

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- *Common Sense (of Sorts): One should only assume that which one cannot test and reject.*

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  - Modest Asymmetry — Left tail is fatter than the right tail

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- Second Stylized Fact: **Asset returns are not independent. At a minimum their squares show substantial predictability**

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- **The stochastic features of asset returns may possess many mysteries, but there are also consistent behaviors that are found across different nations, across different asset classes, and over many different time periods and time scales.**

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- **The Black-Scholes Model is brutally at odds with the most fundamental stylized facts for stock returns. What's up with that?**

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  - Everyone does this to some extent, but there is probably a benefit to being as systematic as one can be.

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  - post October 1987 Black-Scholes is "broken" as a direct guide to market value (but markets continue to grow as new comfort levels of risk allocation are reached)

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