Note on Rates of Return
For Zero-Cost Portfolios (Revised 12/7)

The notion of a "zero-cost portfolio" shows up in many places, but most notably in the Fama-French Four Factor Model and its sequelae. How does on compute the return on such a portfolio?

Naturally, it is a matter of "money in" and "money out," but there is a twist since this will be different for individual versus institutional investors. For the purpose of theory, the return to the institutional investor is more important, but we will look at both.

Individual Investor

If we have two assets A and B with random returns in period t of R_A and R_B then the portfolio of long one dollar of A and short one dollar of B will typically require one dollar of an individual's capital to create because of the margin balance required on the short position. At the end of the period, the one dollar of capital will have "grown" to $1 + R_A - R_B$ dollars. The return is a plain vanilla $R_A - R_B$, this has mean $\mu_A - \mu_B$ and variance $\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B$. In the case of sector returns, $\mu_A - \mu_B$ could easily be less than the short-rate r, so one might not be surprised to find negative Sharpe ratios. Naturally, if B really was a good short, then μ_B would be negative, and the Sharpe ratio could be a whopper.

Institutional Investor using Leverage

The institutional investor will receive an "interest rebate" on the short position, which for illustration we will take to be $3/4$ of the short rate r. The institutional investor will typically be able to take on the "zero-cost portfolio" with one long dollar and one short dollar will less than 1 of committed capital. Let's let c denote the capital required, and c could be as small as $1/10$. For simplicity, we'll also suppose that the institution can borrow at the sort rate r to finance the long position.

Imagine also that the full position is financed — the "desk is fully financed" but relies on (part of) the balance sheet of the firm to make the financing possible. This is not 100% right, but it would get messy to add much more verisimilitude. As a first approximation, this is certainly decent.

Thus, c "committed dollars" grows to $c + R_A - R_B - (1/4)r$ dollars in period t since the net charge for carry on the one-dollar-long one-dollar-short position is $(3/4)r - r = (-1/4)r$. The one period return to the institution for each committed dollar is thus

$$(R_A - R_B - (1/4)r)/c.$$

For this period we have expected returns

$$(\mu_A - \mu_B - (1/4)r)/c$$

and variance

$$(\sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B)/c^2.$$

Institutional Investor Without Leverage

Now suppose one does not use leverage. That is you do not borrow to support your long position, but you do get the rebated on the short position. You put up a dollar to create the "zero cost portfolio". Now you get the rebate but you don't need to pay any margin. In one period, one dollar grows to

$$1 + R_A - R_B + (3/4)r$$

so the rate of return is

$$R_A - R_B + (3/4)r.$$

Life May Be Beautiful ... Yet Not Simple

Please think through what I have written here and pass back any observations that you have. It is easy to be glib about returns, but when both leverage and rebates are involve things can get tricky.