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Lower Bounds for Algebraic Decision Trees

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A topological method is given for obtaining lower bounds for the height of algebraic decision trees. The method is applied to the knapsack problem where an $\Omega(n^2)$ bound is obtained for trees with bounded-degree polynomial tests, thus extending the Dobkin-Lipton result for linear trees. Applications to the convex hull problem and the distinct element problem are also indicated. Some open problems are discussed.

1. INTRODUCTION

Decision trees are often used to model algorithms for combinatorial and geometrical problems. While motivation for these models rests primarily on their generality and conceptual simplicity, they also have the benefit of offering at present the most promising prospect for proving worst-case lower bounds in many problems.

For *linear* decision trees several powerful techniques are known for bounding the tree height from below (e.g., Reingold [11], Dobkin [3], Dobkin and Lipton [4, 5], Yao [15], and Yao and Rivest [17]).

Much less is known for general *algebraic* decision trees. Beyond the naive information bound, Rabin's theorem (Rabin [10]) and the convex hull problem (Yao [16]) are apparently the only known results.

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The purpose of this article is to provide a general method for establishing lower bounds for the worst-case performance of algorithms prescribed by arbitrary algebraic decision trees. Technically this work extends the results of Dobkin and Lipton [4, 5], but the tools put to work here provide nontrivial bounds for a large class of previously untouchable problems.

Before giving the detailed computational model it seems worthwhile to mention informally a concrete application.

THEOREM 1. *Any algebraic decision tree of bounded order which solves the n-dimensional knapsack problem must have height at least* $\Omega(n^2)$.

This result extends the knapsack bounds under the *linear* decision tree model due to Dobkin and Lipton [4] and the $\Omega(n \log n)$ result of Dobkin [3].

The method used here rests critically on a result from real algebraic geometry due to Milnor [9]. Since the machinery used by Milnor may not be familiar to workers in complexity, we have tried to give an exposition of the basic facts necessary for making this work self-contained. The bounds discussed here should prove useful in many related problems.

In the next section we rigorously specify the computational model and outline the lower bound method. The third section exposits Milnor's inequality and gives a heuristic argument which tries to pinpoint the necessity for the more sophisticated tools.

The fourth section is devoted to applications and in particular to the proof of the result on the knapsack problem (Theorem 1) which was mentioned above.

The final section mentions some open problems and suggests a line of attack which if sufficiently developed might add significantly to the power of the present method.

2. Computational Model and the General Method

Let $W \subseteq \mathfrak{R}^n$ be any set. A (dth-*order*) *decision tree* T for testing if $\mathbf{x} \in W$ is a ternary tree with each internal node containing a test of the form $p(x_1, x_2, \ldots, x_n)$: 0, where p is a polynomial of degree at most d. Each leaf of T contains a "yes" or a "no" answer. For an input $\mathbf{x}$, the procedure starts at the root and traverses down the tree. At each internal node a branching is made according to the polynomial test at that node and when a leaf is reached the answer to the question "Is $\mathbf{x} \in W$?" must be given correctly.

Now let $C_d(W)$ be the minimum height h_T for any dth-order decision tree T (for the set W). Our key objective will be to obtain lower bounds on $C_d(W)$, and the bound given here will depend heavily on the topology of W.

By $^{\#}W$ we denote the number of (disjoint) connected components of W. Also for any polynomial $p(x_1, x_2, \ldots, x_n)$ we set $S_p = \{\mathbf{x} | p(\mathbf{x}) \neq 0\}$, and for any integers n, $m > 0$ we put $\beta(m, n) = \max\{{}^{\#}S_p \mid p$ is a polynomial of n real variables and of degree at most $m\}$.

The following elementary result provides the skeleton of our method. (To put flesh on the bones will require the bounds on β obtained in the next section.)

THEOREM 2. *Let $W \subseteq \mathfrak{R}^n$ be an open set, and let T be a dth-order algebraic decision tree for deciding if $\mathbf{x} \in W$. If W is the disjoint union of N open sets, then the height h_T satisfies the inequality*

$$2^{h_T}\beta(h_T d, n) \geq N.$$

Proof. For each leaf l of T let V_l be the set of inputs $\mathbf{x} \in R^n$ leading to l and let I_l be the set of constraints resulting from the tests. Let $\mathfrak{L}$ be the set of leaves l such that I_l consists only of strict inequalities and such that the answer stored at l is "yes." One should note that, for every $l \in \mathfrak{L}$, V_l is an open set and $V_l \subseteq W$.

We now write $W = \bigcup_{i=1}^{N} W_i$ where each W_i is a connected open set and the W_i are disjoint, and write $V_l = \{\mathbf{x}: \bar{p}_{l_1}(\mathbf{x}) < 0, \bar{p}_{l_2}(\mathbf{x}) < 0, \ldots, \bar{p}_{l_s}(\mathbf{x}) < 0\}$ where each $\bar{p}_{l_i}$ is a polynomial of degree not greater than d and where $s \leq h_T$. As a consequence of this representation, $V_l \subseteq \{\mathbf{x} \mid q_l(\mathbf{x}) \neq 0\} = \mathfrak{D}$, where $q_l(\mathbf{x}) = \prod \bar{p}_{l_j}(\mathbf{x})$ is a polynomial of degree at most $h_T d$. Moreover, each connected component of V_l is contained in at most one component of $\mathfrak{D}$. Hence, V_l has at most $\beta(h_T d, n)$ connected components $V_{l1}, V_{l2}, \ldots$.

Since each leaf of T is correctly labeled, each V_{lj} has to be completely contained in some W_i. Since the number of such V_{lj} is at most $\beta(h_T d, n)$ and there are only $|\mathfrak{L}|$ values of l which lead to "yes," the number of components N of W is bounded by $|\mathfrak{L}|\,\beta(h_T d, n)$. In the last deduction, we have used the fact that each W_i must contain some points in $\bigcup_l V_l$ (since the set $W - \bigcup_l V_l$ is of measure 0). Since $2^{h_T} \geq |\mathfrak{L}|$ the theorem follows. $\square$

3. COUNTING CONNECTED COMPONENTS

To use Theorem 2 one needs bounds on $\beta(m, n)$ and this is apparently no easy matter. Fortunately, there is a bound due to Milnor [9, Theorem 3] which is sufficient for some applications:

$$\beta(m, n) \leq (m + 2)(m + 1)^{n-1}. \tag{3.1}$$

The proof of Milnor's inequality rests on the several substantial results from Morse theory and algebraic topology, about which we will say more below. First we would like to give a heuristic indication of an analogous result.

The only preliminary needed for the argument is Bezout's theorem which says that any system of n algebraic equations in n variables with degree d

has either infinitely many (complex) solutions or at most d^n. For a classical approach to the proof of Bezout's theorem one can consult Enriques [7], or, for the case $n = 2$, there is a nice proof in Seidenberg [12].

To use Bezout's theorem we suppose that p is a real polynomial in n variables with degree m, and we note that R can be chosen so that $A = \{\mathbf{x} \mid \mathbf{x} \in R^n,\ p(\mathbf{x}) > 0,\ R^2 - \sum_{i=1}^n x_i^2 > 0\}$ has at least as many *bounded* connected components as $\{\mathbf{x} \mid p(\mathbf{x}) > 0\}$ has connected components (bounded or unbounded). Since each bounded connected component of A must contain a local maximum of pq, the number of bounded components of A is majorized by the number of zeros of the system $\nabla pq = 0$. By Bezout's theorem, this number is either infinite, or else bounded by $(m + 1)^n$.

This finite bound is for our purposes almost as sharp as Milnor's bound. The real work comes in providing a rigorous perturbation argument which rules out the case when Bezout gives only the trivial infinite bound. That is precisely the case which causes all the trouble and presents this section from being self contained.

The inequalities proved in Milnor [9] actually provide bounds on the sum of the Betti numbers (in Čech cohomology) for the set $V = \{\mathbf{x} | \mathbf{x} \in R^n,\ p(\mathbf{x}) \geq 0\}$, where p is a polynomial of degree at most m. This leads immediately to the bound ${}^{\#}V \leq \frac{1}{2}(m + 2)(m + 1)^{n-1}$, since the zeroth Betti number is equal to the number of connected components (see Eilenberg and Steenrod [6, p. 254, Exercise A.3]). The same bound remains true when V is defined by a strict inequality $p(\mathbf{x}) > 0$; one can see this by utilizing the fact $\{\mathbf{x} | p(\mathbf{x}) > 0\} = \cup_{\epsilon > 0} V_\epsilon$, where $V_\epsilon = \{\mathbf{x} | p(\mathbf{x}) \geq \epsilon\}$. This implies inequality (3.1).

For readers who are not familiar with the notions of Homology groups and Betti numbers, Chapter 6 of Hocking and Young [8] can provide a pleasant introduction. We should further remark that a recent exposition of Bott [2] gives a direct and intuitive introduction to Milnor [9] and the closely related ideas of Morse theory. As it happens, the problem of determining $\beta(m, n)$ is probably quite deep. As noted in Arnold [1], it is intimately connected with the basic concerns of Hilbert's 16th Problem.

Despite the apparent simplicity of the heuristic argument given above, we benefit from a genuine stroke of good luck that any bound like (3.1) is known.

4. Applications

We now use Theorem 2 to derive lower bounds. Clearly, the function $2^x\beta(xd, n)$ is an increasing function of x. Let $\alpha(d, n, N)$ be the minimum x satisfying $2^x\beta(xd, n) \geq N$. Theorem 2 immediately yields the following formal bounds:

$$C_d(W) \geq \alpha(d, n, {}^{\#}W). \tag{4.1}$$

Any general upper bounds on β can be used to derive lower bounds on α and hence C_d. In particular, Milnor's bound (3.1) gives the following result.

THEOREM 3. *For any real* ϵ,

$$C_d(W) \geq \min\left\{\epsilon \log_2 N, \frac{1}{d}\left(N^{(1-\epsilon)/n} - 1\right)\right\},$$

when $N = {}^{\#}W$.

COROLLARY. *If* ${}^{\#}W = \Omega(n^{(1+\delta)n})$ *for some fixed* $\delta > 0$, *then*

$$C_d(W) = \Omega\big(\log({}^{\#}W)\big).$$

Proof. Let $x = C_d(W)$. Then $2^x\beta(xd, n) \geq N$. Hence by (3.1)

$$2^x(xd+1)^n \geq N.$$

Either $2^x \geq N^\epsilon$ or $(xd+1)^n \geq N^{1-\epsilon}$, proving the theorem. □

The corollary follows by writing ${}^{\#}W = n^{n(1+g(n))}$ and setting

$$\epsilon = \frac{1}{2}\,\frac{g(n)}{1+g(n)}$$

in the theorem.

Thus, Theorem 3 gives a lower bound nonlinear in n when ${}^{\#}W$ grows at least as fast as $n^{(1+\delta)n}$. This is also necessary since the theorem only gives a lower bound $O(n)$ when ${}^{\#}W = O(n^n)$.

In the first example given below, ${}^{\#}W \approx 2^{n^2/2}$; thus we have a good lower bound. The other two examples have ${}^{\#}W \leq n^n$, and Theorem 3 does not give nonlinear bounds. However, Theorem 2 (or, (4.1)) is still true for these later examples, and a better determination in the future may result in an improved bound.

EXAMPLE 1. THE KNAPSACK PROBLEM. Given real numbers $x_1, x_2, \ldots, x_n$, decide if there exists some subset $S \subseteq \{1, 2, \ldots, n\}$ such that $\Sigma_{i\in S} x_i = 1$.

In this case, $W = \{(x_1, x_2, \ldots, x_n) | \Pi_S(\Sigma_{i\in S} x_i - 1) \neq 0\}$. It was shown in Dobkin and Lipton [4] that ${}^{\#}W \geq 2^{n^2/2}$. Thus, $C_d(W) = \Omega(n^2)$ for any fixed d. This generalizes the result of Dobkin and Lipton where they showed $C_1(W) = \Omega(n^2)$.

EXAMPLE 2. ELEMENT DISTINCTNESS. Given $x_1, x_2, \ldots, x_n \in \mathfrak{R}$, is there a pair i, j with $i \neq j$ and $x_i = x_j$? In this case,

$$W = \left\{ (x_1, x_2, \ldots, x_n) \,\middle|\, \prod_{i \neq j} (x_i - x_j) \neq 0 \right\} \subset \mathfrak{R}^n.$$

It is easily shown that $^{\#}W = n!$ since each region $\{(x_1, x_2, \ldots, x_n) \mid x_{\sigma(1)} < x_{\sigma(2)} < \ldots x_{\sigma(n)}\}$ is a maximal connected component of W for each permutation σ. One therefore has $C_d(W) \geq \alpha(d, n, n!)$.

EXAMPLE 3. EXTREME POINTS. Given n points on the plane does the convex hull formed by them possess n vertices?

Here W cannot be expressed by an easy algebraic relation but it is still possible to show $^{\#}W \geq (n-1)!$. Obviously, W is an open set in $(\mathfrak{R}^2)^n$. For any configuration $\{x_1, x_2, \ldots, x_n\}$ in $W \subset (\mathfrak{R}^2)^n$, we have a cyclical ordering σ of the points $\{x_i \mid 1 \leq i \leq n\}$ which is given uniquely by taking the points in cyclical order. Clearly, any of the $(n-1)!$ cyclical permutations can arise in this way so all that remains is to show that if $\sigma \neq \sigma'$ then the configurations which give rise to these permutations are in disjoint components of W.

For each configuration in W we consider the $\binom{n}{3}$ element array A given by $\Delta(x_i x_j x_k)$, where Δ is the signed area of the triangle formed by the 3-set $\{x_i x_j x_k\} \subset \{x_1, x_2, \ldots, x_n\}$. If the configuration corresponding to σ is continuously deformed in any way to the configuration for σ' then A_σ is transformed continuously into $A_{\sigma'}$. Since σ and σ' differ there is some triple $\{x_i x_j x_k\}$ for which $\Delta(x_i x_j x_k)$ has differing signs in A_σ and $A_{\sigma'}$. By the intermediate value theorem there is therefore some time during the continuous deformation when $\Delta(x_i x_j x_k) = 0$. This says that x_i, x_j, x_k are then collinear and at that point there are at most $n-1$ extreme points in the configuration. This proves that any passage from σ to σ' must go out of W, so σ and σ' correspond to different components.

The main consequence of the preceding bound is that

$$C_d(W) \geq \alpha(d, 2n, (n-1)!)$$

and it was originally hoped that this would be sufficient to prove a conjecture of Yao [16] that any algebraic decision tree of order d for the extreme point problem must have height $\Omega(n \log n)$. The Milnor bound in this case is not sufficiently sharp to obtain the desired bound. We indicate in the next section a bound which would be sufficient.

While these last two examples are disappointing in that they do not give the conjectured nonlinear lower bounds, one should note that since only a yes–no answer is required there is a logical necessity of only two terminal leaves. So, the information theoretic bound in these two cases gives only the absurd bound $\log_2 2$.

5. Open Problems and Directions

Surely the most interesting and important problems pivot about finding sharper bounds on $\beta(m, n)$. It is conceivable that $\beta(m, n) = 2^{O(m+n)}$ which could imply by Theorem 2 that $C_d(W) = \Omega(1/d(\log_2 N - m))$. This bound would yield an $\Omega(n \log n)$ lower bound in Examples 2 and 3 for fixed d.

In fact, a somewhat weaker result will suffice for this purpose. Let $\beta(d, m', n)$ be the maximum of ${}^{\#}S_p$ for any p of the form $\prod_{i=1}^{m'} p_i(x_1, x_2, \ldots, x_n)$, with each p_i of degree not greater than d. Clearly $\beta(d, m', n) \leq \beta(dm', n)$. The result one really needs in Examples 2 and 3 is $\beta(d, m', n) = 2^{O(dm'+n)}$. Can one prove better bounds on $\beta(d, m', n)$ than on $\beta(m, n)$? Here we note that it is not hard to see that

$$\beta(1, m', n) \leq \sum_{j=0}^{n} \binom{m'}{j}, \qquad n \leq m', \tag{5.1}$$

since $\beta(1, m', n)$ just equals the number of regions of $\mathfrak{R}^n$ which can be partitioned by m' hyperplanes. (This is proved in Steiner (1926) [13], which is in the first volume of Crelle's *J. Reine Angew. Math.* and which is better remembered for containing five fundamental papers of N. H. Abel. For modern treatment of (5.1) see Wetzel [14] and the references given there.)

A more modest approach to the problems suggested by Examples 2 and 3 rests on obtaining bounds for any small values of $d \geq 2$. It is known (Yao [16]) that $C_2(W) = \Omega(n \log n)$ in Example 3, but there are no other known nonlinear lower bounds even in the case $d = 3$.

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