REFINEMENT OF HARDY'S INEQUALITY

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Abstract. We give a simple combinatorial proof of an inequality that refines Hardy's inequality. As a corollary we obtain a corresponding refinement of Carleman's inequality.

1. Introduction

Given real numbers a_j, $1 \le j \le n$, we let $A_j = (a_1 + a_2 + \cdots + a_j)/j$ and consider the $n \times n$ matrix $C = \{c_{jk}\}$ defined by setting

$$c_{jk} = \begin{cases} a_k & \text{for} \quad 1 \le j < k \le n \\ A_j & \text{for} \quad 1 \le k \le j \le n. \end{cases} \tag{1}$$

Thus, for example, in the 3×3 case we have

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} A_1 & a_2 & a_3 \\ A_2 & A_2 & a_3 \\ A_3 & A_3 & A_3 \end{pmatrix}.$$

The main purpose of this article is to prove the following theorem together with some extensions and related results.

Theorem 1. *For any mapping $\sigma : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ one has*

$$\sum_{k=1}^{n} c^2_{\sigma(k),k} \le 4 \sum_{k=1}^{n} a_k^2. \tag{2}$$

By taking $\sigma(k) = k$ for all $1 \le k \le n$ one recovers Hardy's inequality in its simplest form [4, p.169],

$$\sum_{k=1}^{n} A_k^2 \le 4 \sum_{k=1}^{n} a_k^2. \tag{3}$$

but other choices of σ can lead to novel results. For example, one has the following complement of Hardy's inequality which is typically sharper.

Corollary 1.

$$\frac{1}{n} a_1^2 + \left(1 + \frac{1}{n}\right) \sum_{k=2}^{n} A_k^2 \le 4 \sum_{k=1}^{n} a_k^2. \tag{4}$$

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To get the inequality (4) from Theorem 1, we again take $\sigma(k) = k$ for $2 \le k \le n$, but this time we choose $\sigma(1) = s$ where $c_{s,1}^2 = \max_j c_{j,1}^2$. For such a σ then have the trivial bounds

$$(A_1^2 + A_2^2 + \cdots + A_n^2)/n \le c_{\sigma(1),1}^2 \quad \text{and} \quad A_k^2 = c_{\sigma(k),k}^2 \quad \text{for } k \ge 2,$$

and, when we sum these inequalities and apply (2), we obtain the bound (4).

2. Proof of Theorem 2

For each $0 \le x < \infty$, we consider the disjoint sets $A(j,x) \subset \{1, 2, ..., n\}$ that are defined by setting

$$A(1,x) = \{k : |c_{1k}| \ge x\} \quad \text{and}$$
$$A(j,x) = \{k : \max_{1 \le i \le j-1} |c_{ik}| < x \le |c_{jk}|\} \quad \text{for} \quad 2 \le j \le n.$$

The key observation is that one can prove the bound

$$\sum_{k \in A(j,x)} |c_{jk}| \le \sum_{k \in A(j,x)} |a_k|. \tag{5}$$

To see why (5) holds, we first note that $A(j,x)$ is either a (possibly empty) subset S of the set $\{j+1, j+2, ..., n\}$, or else it is a set of the form $S \cup I$ where S is a subset of $\{j+1, j+2, ..., n\}$ and $I = \{1, 2, ..., j\}$.

In the first case, the inequality (5) is trivial since one has $c_{jk} = a_k$ whenever j is less than k. In the second case, one only needs the further observation that $c_{jk} = A_j$ for all $k \in I$, so we have

$$\sum_{k \in I} |c_{jk}| = j|A_j| \le \sum_{k \in I} |a_k|.$$

Taken together, these two observations prove (5).

By the definition of $A(j,x)$ we have $x \le |c_{jk}|$ for each $k \in A(j,x)$, so if we write $\mathbb{I}(\cdot)$ for the indicator function, then our basic bound (5) gives us

$$\sum_{k=1}^{n} x\mathbb{I}(k \in A(j,x)) \le \sum_{k=1}^{n} |a_k|\mathbb{I}(k \in A(j,x)). \tag{6}$$

Since the sets $A(j,x)$, $1 \le j \le n$ are disjoint, if we sum the bound (6), and set $B(x) = \cup_j A(x,j)$, then we have

$$\sum_{k=1}^{n} x\mathbb{I}(k \in B(x)) \le \sum_{k=1}^{n} |a_k|\mathbb{I}(k \in B(x)).$$

Now, if we let $m_k = \max_{1 \le j \le n} |c_{jk}|$, then $k \in B(x)$ if and only if $m_k \ge x$, so we can rewrite the preceding sums as

$$\sum_{k=1}^{n} x\mathbb{I}(x \le m_k) \le \sum_{k=1}^{n} |a_k|\mathbb{I}(x \le m_k). \tag{7}$$

If we integrate over $x \in [0, \infty)$ and apply Cauchy's inequality, then (7) yields

$$\frac{1}{2}\sum_{k=1}^{n} m_k^2 \le \sum_{k=1}^{n} |a_k| m_k \le \left(\sum_{k=1}^{n} |a_k|^2\right)^{1/2} \left(\sum_{k=1}^{n} m_k^2\right)^{1/2}, \tag{8}$$

and, when we square this and simplify, we obtain

$$\sum_{k=1}^{n} m_k^2 \le 4 \sum_{k=1}^{n} |a_k|^2,$$

which is equivalent to the inequality (2) of Theorem 1.

3. Methods and Extensions

From inequality (7) onward, our proof simply parallels the classic argument for Doob's maximal inequality for submartingales (say, as given by Kallenberg [2, p.106]). The main point here has been to suggest how in the case of the refined Hardy inequality (2) one can dispense with any explicit use of martingale theory. The key inequality (5) is completely elementary, and, from that point on, the proof follows a path that would have been familiar even in Hardy's day.

Just as in the theory of martingales, one can use the elementary bound (7) to prove an ℓ^p inequality for any $p > 1$. Specifically, if one multiplies (7) by px^{p-2}, integrates, and then applies Hölder's inequality one finds

$$\sum_{k=1}^{n} m_k^p \le \frac{p}{p-1} \sum_{k=1}^{n} |a_k| m_k^{p-1} \le \frac{p}{p-1} \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} \left(\sum_{k=1}^{n} m_k^p \right)^{(p-1)/p}$$

which, after some simplification, gives us

Theorem 2. *For any mapping $\sigma : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ and any $p > 1$, one has the bound*

$$\left(\sum_{k=1}^{n} |c_{\sigma(k),k}|^p \right)^{1/p} \le \frac{p}{p-1} \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p}. \tag{9}$$

4. A Refinement of Carleman's Inequality

Hardy, Littlewood, and Pólya [1, p. 249] observed that the classical ℓ^p Hardy inequality can be used to prove Carleman's inequality, and their argument can also be combined with the inequality (9) to obtain a corresponding refinement of Carleman's inequality. Specifically, by replacing a_j with $|a_j|^{1/p}$ in the bound (9), taking the p-power, and letting $p \to \infty$, one finds the following

Theorem 3. *For all $\sigma : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ one has*

$$\sum_{k=1}^{n} b_{\sigma(k),k} \le e \sum_{k=1}^{n} |a_k|, \tag{10}$$

where $e = 2.71828...$ is the natural base and where

$$b_{jk} = \begin{cases} |a_k| & \text{for } 1 \le j < k \le n \\ |a_1 a_2 \cdots a_j|^{1/j} & \text{for } 1 \le k \le j \le n. \end{cases}$$

If one takes $\sigma(k) = k$, then Theorem 3 recovers the classical Carleman inequality,

$$\sum_{k=1}^{n} |a_1 a_2 \cdots a_k|^{1/k} \le e \sum_{k=1}^{n} |a_k|,$$

but, as in the case of Corollary 1, one can do a bit better by an appropriate choice of σ. For example, if we set $\sigma(k) = k$ for $2 \le k \le n$ and take $\sigma(1) = s$ where

$b_{s,1} = \max_j b_{j,1}$, then, parallel to the proof of Corollary 1, one finds that the bound (10) gives a slightly refined version of Carleman's inequality.

Corollary 2. *For all real or complex $a_1, a_2, ..., a_n$, one has*

$$\frac{1}{n}|a_1| + \left(1 + \frac{1}{n}\right) \sum_{k=2}^{n} |a_1 a_2 \cdots a_j|^{1/j} \le e \sum_{k=1}^{n} |a_k|.$$

5. Rearrangements

If we introduce the difference

$$D(a_1, a_2, ..., a_n) = 4 \sum_{k=1}^{n} a_k^2 - \sum_{k=1}^{n} \max_{1 \le j \le n} c_{j,k}^2$$

then Theorem 2 tells us that $D(a_1, a_2, ..., a_n) \ge 0$ for all real sequences, and it is natural to ask how $D(a_1, a_2, ..., a_n)$ might depend on the order of the sequence. In particular, given a nonnegative sequence $a_1, a_2, ..., a_n$ with the monotone rearrangements

$$a_1^{\downarrow} \ge a_2^{\downarrow} \ge \cdots \ge a_n^{\downarrow} \quad \text{and} \quad a_1^{\uparrow} \le a_2^{\uparrow} \le \cdots \le a_n^{\uparrow},$$

the one has the following.

Theorem 4. *For nonnegative sequence $a_1, a_2, ..., a_n$ one has the bounds*

$$D(a_1^{\downarrow}, a_2^{\downarrow}, \ldots, a_n^{\downarrow}) \le D(a_1, a_2, ..., a_n) \le D(a_1^{\uparrow}, a_2^{\uparrow}, \ldots, a_n^{\uparrow}). \tag{11}$$

In other words, the bound (2) is sharpest when $a_1, a_2, ..., a_n$ is in nonincreasing order and loosest when $a_1, a_2, ..., a_n$ is in nondecreasing order.

It is natural to try to prove these bounds with an interchange argument. If the vector $\mathbf{a} = (a_1, a_2, ..., a_n)$ is not already in nonincreasing order, then there is a smallest integer m such that $a_m < a_{m+1}$, and we can consider how the value of

$$M(\mathbf{a}) = \sum_{k=1}^{n} \max_{1 \le j \le n} c_{j,k}^2$$

is changed when $\mathbf{a}$ is replaced by $\mathbf{a}' = (a_1, a_2, ..., a_{m+1}, a_m, ..., a_n)$.

If $C' = \{c'_{jk}\}$ denotes the matrix corresponding to $\mathbf{a}'$ via the definition (1), then the matrices C and C' are equal except possibly in parts of row m and columns m and $(m+1)$. If we write $a_m = \alpha$ and $a_{m+1} = \alpha + x$, then the first m values in row m of C' are all equal to $((m-1)A_{m-1} + a + x)/m$, and the corresponding values of C are equal to $((m-1)A_{m-1} + a)/m$. This implies that the first $m-1$ summands of $M(\mathbf{a}')$ are not smaller than the corresponding summands of $M(\mathbf{a})$.

From the definition (1) we also see that $c_{j,k} = c'_{j,k}$ for all j and k such that $k > m+1$, so, to prove that $M(\mathbf{a}) \le M(\mathbf{a}')$, it suffices to show that $L \le R$ where

$$L = \max_{1 \le j \le n} c_{j,m}^2 + \max_{1 \le j \le n} c_{j,m+1}^2 \quad \text{and} \quad R = \max_{1 \le j \le n} (c'_{j,m})^2 + \max_{1 \le j \le n} (c'_{j,m+1})^2,$$

and these can be rewritten in terms of α, x and A_{m-1} as

$$L = \max\{\alpha, ((m-1)A_{m-1} + \alpha)/m\}^2 + \max\{\alpha + x, ((m-1)A_{m-1} + 2\alpha + x)/m\}^2,$$

$$R = \max\{\alpha, ((m-1)A_{m-1} + \alpha + x)/m\}^2 + \max\{\alpha, ((m-1)A_{m-1} + 2\alpha + x)/m\}^2.$$

To prove $L \le R$ one just needs to consider three cases,

$$(i) \quad \alpha + x \le A_{m-1}, \quad (ii) \quad \alpha \le A_{m-1} < \alpha + x \quad \text{and} \quad (ii) \quad \alpha < A_{m-1},$$

and in each instance one can easily check that $L \leq R$. This calculation shows that $M(\mathbf{a}) \leq M(\mathbf{a}')$, and by repeating the argument one finds $M(\mathbf{a}) \leq M(\mathbf{a}^{\downarrow})$ where $\mathbf{a}^{\downarrow} = (a_1^{\downarrow}, a_2^{\downarrow}, \ldots, a_n^{\downarrow})$. Since the first summand of $D(\mathbf{a})$ is unchanged by rearrangements, this proves the first inequality of (11); the second inequality is obtained analogously.

6. Final Remark

Hardy's original motivation for the inequality (3) was to give an elementary proof of Hilbert's inequality,

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m a_n}{m+n} \leq \pi \sum_{n=1}^{\infty} a_n^2, \tag{12}$$

and, as explained in [3], Hardy's inequality (3) does yield analogs of Hilbert's bound (12). Some of these can be modestly refined by using (2) in place of (3).

References

[1] Hardy G.H., Littlewood, J.E., and Pólya, G. (1952). *Inequalities*, 2nd Edition, Cambridge University Press, Cambridge, UK.

[2] Kallenberg, O. (1995). *Foundations of Modern Probability*, Springer, New York.

[3] Kufner, A., Maligranda, L., and Persson, L.-E. (2005). The Prehistory of the Hardy Inequality, *The American Mathematical Monthly*, to appear.

[4] Steele, J.M. (2004). *The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*, Cambridge University Press, Cambridge.

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