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PROBABILISTIC ALGORITHM FOR THE DIRECTED TRAVELING SALESMAN PROBLEM*†

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A model is given for a random directed traveling salesman problem (DTSP). The asymptotic behavior of the optimal solution of the DTSP is determined, and this result is used to establish an ϵ-optimal probabilistic algorithm for solving the DTSP in polynomial time.

1. Introduction. In Karp (1977) the problem is posed of formulating a probabilistic model of the *directed* traveling salesman problem (DTSP) for which one can establish a probabilistic polynomial time algorithm. The main objective of this article is to introduce one such model.

In fact, the model studied here is about the most obvious model one could imagine for the DTSP. The hard part is to obtain enough probabilistic information from the model to be able to show the existence of a good algorithm.

To specify the model we first suppose that X_i, $1 \leqslant i < \infty$, are independent random variables with the uniform distribution in the unit square $[0, 1]^2$. As the vertex set of a directed graph G_n in $\mathbb{R}^2$ we take $V_n = \{X_1, X_2, \ldots, X_n\}$. Now, we suppose that for $1 \leqslant i < j \leqslant n$ there are independent Bernoulli random variables Y_{ij} which are also independent of V_n and for which $P(Y_{ij} = 1) = 1/2 = 1 - P(Y_{ij} = 0)$. The directed edge set E_n is defined by taking $(X_i, X_j) \in E_n$ if $Y_{ij} = 1$ and $(X_j, X_i) \in E_n$ if $Y_{ij} = 0$. The random variable of greatest interest is D_n, the length (in the usual Euclidean distance) of the shortest legitimate directed path through all of the vertices V_n of the graph G_n.

It may not be apparent that there is always a directed path through V_n. This follows from a classic result of Rédei (1934) and will be established algorithmically in the next section.

The main result on D_n which will be proved here is the following:

THEOREM 1. *There is a constant* $0 < \beta < \infty$ *such that as* $n \to \infty$

$$ED_n \sim \beta\sqrt{n}\,. \tag{1.1}$$

The main consequence of this asymptotic relationship is the existence of a probabilistically efficient algorithm for the DTSP.

THEOREM 2. *There is a polynomial time algorithm which provides a directed path through* V_n *which has length* D_n^* *satisfying*

$$ED_n^* \leqslant (1 + \epsilon)ED_n, \tag{1.2}$$

for all $\epsilon > 0$ *and* $n \geqslant N(\epsilon)$.

The sense of optimality in this result is a bit weaker than that obtained by Karp (1976), (1977), and the reasons for this difference are discussed in the final section.

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The proof of Theorem 1 is given in the next three sections. In the first of these, a procedure is given for sewing together the subproblems of a natural decomposition of the DTSP. The inequalities provided by this procedure are used in §3 to obtain the asymptotic behavior of the Borel average of the ED_n. Finally, the Tauberian theorem of R. Schmidt (1925) is used in §4 to complete the proof of Theorem 1.

In §5 the sewing method and Theorem 1 are used to complete the proof of Theorem 2. The last section compares results obtained here for the DTSP with Karp's original work on the TSP and isolates a basic open problem.

TECHNICAL REMARKS. (1) By a directed path we mean a sequence of directed edges $e_1, e_2, \ldots, e_n$ such that if $e_i = (x_i, y_i)$ then $x_{i+1} = y_i$ for each $1 \leqslant i < n$. In particular it is possible for a vertex x to appear more than once on the shortest path. This is a complication that cannot arise in the undirected TSP.

(2) The fact that $\beta > 0$ is a consequence of the fact that the length of the DTSP is at least as large as the corresponding TSP. Thus, $\beta > 0$ follows from Beardwood, Halton, and Hammersley (1958). To see this even more easily one can note that the expected distance from any point in $\{X_1, X_2, \ldots, X_n\}$ to its nearest neighbor is bounded below by $cn^{-1/2}$.

2. Sewing inequalities.

As promised, we will first establish Rédei's theorem that any complete digraph G has a directed path through all its vertices. Suppose a partial path $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$, $1 \leqslant k < n$, has been constructed through part of the vertex set $\{x_1, x_2, \ldots, x_n\}$. We can choose x_j arbitrarily from the remaining vertices and show that it can be included in an augmented path. If $(x_j, x_{i_1}) \in E$ then $x_j \to x_{i_1} \to \cdots \to x_{i_k}$ is such a path, and otherwise $(x_{i_1}, x_j) \in E$ by the completeness of G. Now, if $(x_j, x_{i_2}) \in E$ we see $x_{i_1} \to x_j \to x_{i_2} \to \cdots \to x_{i_k}$ is a path, while if $(x_j, x_{i_2}) \notin E$ we proceed to x_{i_3}. Either we eventually succeed in inserting x_j somewhere inside the path or else we have shown that $x_{i_1} \to x_{i_2} \to \cdots \to x_{i_k} \to x_j$ is a legitimate path. This procedure is basic to the inequalities proved here and will be used repeatedly in the sequel.

Now for some probabilistic considerations. Instead of directly studying X_i, $1 \leqslant i < \infty$, i.i.d. $U[0,1]^2$, it will be convenient to consider a Poisson process Π in $\mathbb{R}^2$ with constant intensity 1. For each Borel $A \subset \mathbb{R}^2$ we note that $\Pi(A)$ is a finite point set with cardinality $N_A = |\Pi(A)|$ where N_A is a Poisson random variable with mean $\lambda(A)$, the Lebesgue measure of A. We let $\mathbb{R}^2$ be ordered by Lexico-graphical order ($\ll$), and for each pair of points $x \ll y$ we define a Bernoulli random variable Y_{xy}. We require (1) $P(Y_{xy} = 1) = 1/2 = P(Y_{xy} = 0)$, (2) the collection $\mathscr{S} = \{Y_{xy} : x \ll y,\ x, y \in \mathbb{R}^2\}$ an independent collection, and (3) the collection $\mathscr{S}$ is independent of Π.

A new process $D(t)$ is defined as the length of the shortest path through the points of $\Pi[0,t]^2$ using only those edges (x, y) for which $x \ll y$ and $Y_{xy} = 1$, or $y \ll x$ and $Y_{xy} = 0$. By $E(t)$ we will denote the set of such legitimate directed edges between the points of $\Pi[0,t]^2$.

The point of introducing $D(t)$ is the fact that it is a sort of smoothed version of D_n. This is made explicit, and useful, by the key identity

$$ED(t) = \sum_{n=2}^{\infty} t(ED_n)e^{-t^2}t^{2n}/n!. \tag{2.1}$$

To see why (2.1) holds we note that conditionally on the event $\{|\Pi([0,t]^2)| = n\}$, the random variables $D(t)$ and tD_n have the same distribution. Since $|\Pi([0,t]^2)|$ is Poisson with mean t^2, (2.1) just expresses the identity $ED(t) = E(E(D(t)) \mid |\Pi([0,t]^2)|)$.

We now let $\phi(t) = ED(t)$ and proceed to prove the first of the sewing inequalities which are needed to obtain the asymptotics of $\phi(t)$.

LEMMA 2.1. *There is a constant $c > 0$ such that*

$$\phi(2t) \leqslant 4\phi(t) + ct, \qquad 0 \leqslant t < \infty. \tag{2.2}$$

PROOF. Let Q_i, $1 \leq i \leq 4$ denote the four quadrants of the square $[0, 2t]^2$, and suppose that $x_1 \to x_2 \to \cdots \to x_u$ and $x_i' \to x_2' \to \cdots \to x_v'$ are the optimal directed paths through the sets $\Pi(Q_i)$ and $\Pi(Q_2)$ respectively. Next, we let

$$\tau = \{\min k : (x_u, x_k') \in E(2t),\ 1 \leq k \leq v\}$$

and let $\tau = \infty$ if $(x_u, x_k') \notin E(2t)$ for all $1 \leq k \leq v$ or if $|\Pi(Q_2)| = 0$. Similarly, we let

$$\tau' = \min\{k : (x_v', x_k) \in E(2t),\ 1 \leq k \leq u\}$$

and $\tau = \infty$ if $(x_v', x_k) \notin E(2t)$ for all $1 \leq k \leq u$ or if $|\Pi(Q_1)| = 0$.

Now if $\tau < \infty$ and $\tau' < \infty$ we see that

$$x_u \to x_\tau' \to x_{\tau+1}' \to \cdots \to x_v' \to x_{\tau'} \to x_{\tau'+1} \to \cdots \to x_{u-1} \to x_u \tag{2.3}$$

is a *circuit* C_{12} which goes through all the points of $\Pi(Q_1) \cup \Pi(Q_2)$ except for at most the set $Z_{12} = \{x_k',\ 1 \leq k < \tau\} \cup \{x_k,\ 1 \leq k < \tau'\}$. (See Figure 1(a).) In a similar fashion we construct a circuit C_{34} through all the points of $\Pi(Q_3) \cup \Pi(Q_4)$ except for a set Z_{34}. For the last step in our construction we pick $x \in C_{12}$, $y \in C_{34}$ and apply Rédei's algorithm to obtain a directed path P through $\{x, y\} \cup Z_{12} \cup Z_{34}$ which visits each point at most once. Finally, we describe a (suboptimal) directed path through $\Pi[0, 2t]^2$. Without loss of generality we may suppose x comes before y on P. For our suboptimal path we take P until we get to x then we take the circuit C_{12} back around to x, then we take P until y, make the circuit C_{34} back to y, and finally finish off the path P. (See Figure 1(b).)

We now need to estimate the expected length of this suboptimal path through $\Pi[0, 2t]^2$. First we bound $L(P)$, the length of the path P, by noting that no edge of P is longer than $t2\sqrt{2}$; and there are exactly $2 + |Z_{12}| + |Z_{34}|$ vertices on P. This provides the bound,

$$EL(P) \leq t2\sqrt{2}\,\{1 + E|Z_{12}| + E|Z_{34}|\} \leq 16t \tag{2.4}$$

where we have used the fact that $|Z_{12}|$ is majorized by a geometric random variable with parameter $p = 1/2$ so $E|Z_{12}| = E|Z_{34}| = 2$.

For the length of C_{12}, $L(C_{12})$, we note

$$L(C_{12}) \leq \sum_{i=1}^{u-1} |x_i - x_{i+1}| + \sum_{i=1}^{v-1} |x_i' - x_{i+1}'| + |x_u - x_\tau'| + |x_u' - x_{\tau'}|. \tag{2.5}$$

Since $|x_u - x_\tau'|$ and $|x_v' - x_{\tau'}|$ are less than $t\sqrt{3}$, taking expectations in (2.5) gives

$$EL(C_{12}) \leq 2\phi(t) + t2\sqrt{3}\,. \tag{2.6}$$

Naturally we obtain the same inequality for C_{34} since $L(C_{34}) \overset{d}{=} L(C_{12})$.

From (2.4), (2.6), and the fact that the path we have constructed is suboptimal we

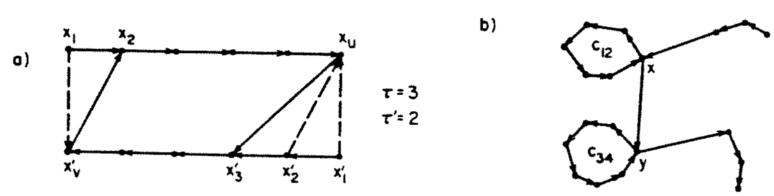

FIGURE 1. Illustration of Lemma 2.1.

have the bound

$$\begin{aligned}\phi(2t) &\leqslant EL(C_{12}) + EL(C_{34}) + EL(P)\\ &\leqslant 4\phi(t) + 30t. \quad \blacksquare\end{aligned} \tag{2.7}$$

LEMMA 2.2. *There is a constant $c > 0$ such that*

$$\phi(3t) \leqslant 9\phi(t) + ct, \qquad 0 \leqslant t < \infty. \tag{2.8}$$

PROOF. The proof is essentially that used before except that the fact that 9 is odd forces some asymmetry. ∎

Before applying these lemmas to the asymptotics of $\phi(t)$, it is worth summarizing a consequence of the sewing procedure which will be useful in the algorithm of §5. We state this as a lemma.

LEMMA 2.3. *Suppose that $\tilde{G}_m$ is a complete digraph with arbitrary vertex set $\{x_1, x_2, \ldots, x_m\} \subset [0,s]^2$ and with the directions of the edge set determined by independent Bernoulli random variables. Let Q_i, $1 \leqslant i \leqslant 4$, denote the four quadrants of $[0,s]^2$ and let D_i denote optimal (or suboptimal) DTSP tours for the restricted digraph $\tilde{G}_i$ with vertex set $Q_i \cap \{x_1, x_2, \ldots, x_m\}$. If D is the solution of the DTSP for G we have*

$$ED \leqslant ED_1 + ED_2 + ED_3 + ED_4 + cs. \tag{2.9}$$

PROOF. This lemma just spells out the consequences of the procedure of Lemma 2.1 in the case of a slightly different model. In particular one should note that in Lemma 2.1 no use was made of the distribution of the points of $\Pi[0,1]^2$. ∎

3. Asymptotics from inequalities.

LEMMA 3.1. *Suppose that $\psi: [0,\infty] \to [0,\infty)$ is any continuous function which satisfies*

$$\psi(2t) \leqslant 4\psi(t) + ct \quad \text{and} \tag{3.1}$$

$$\psi(3t) \leqslant 9\psi(t) + ct, \qquad 0 \leqslant t \leqslant \infty. \tag{3.2}$$

One then has

$$\lim_{t\to\infty} \psi(t)/t^2 = \liminf_{t\to\infty} \psi(t)/t^2 = \beta < \infty. \tag{3.3}$$

PROOF. For $j = 1$, $k = 1$, it is trivial from (3.1) and (3.2) that

$$\psi(2^j t) \leqslant 2^{2j}\psi(t) + ct \sum_{j-1 \leqslant s \leqslant 2(j-1)} 2^s \quad \text{and} \tag{3.4}$$

$$\psi(3^k t) \leqslant 3^{2k}\psi(t) + ct \sum_{k-1 \leqslant s \leqslant 2(k-1)} 3^s. \tag{3.5}$$

These more general inequalities (3.4) and (3.5) are easily verified by induction. Using both of these we see

$$\begin{aligned}\psi(2^j 3^k t) &\leqslant 2^{2j}\psi(3^k t) + ct3^k \sum_{j-1 \leqslant s \leqslant 2(j-1)} 2^s\\ &\leqslant 2^{2j}\left\{3^{2k}\psi(t) + ct \sum_{k-1 \leqslant s \leqslant 2(k-1)} 3^s\right\} + ct3^k \sum_{j-1 \leqslant s \leqslant 2(j-1)} 2^s\\ &\leqslant (2^j 3^k)^2\psi(t) + (2^j 3^k)^2 ct.\end{aligned} \tag{3.6}$$

Now consider $\{n_1 < n_2 < \cdots\} = \{2^j 3^k : j \geqslant 0, k \geqslant 0\} = S$. For any $\epsilon > 0$ there exist positive integers a, b, c, d with $1 \leqslant 2^a 3^{-b} \leqslant 1 + \epsilon$ and $1 \leqslant 3^c 2^{-d} \leqslant 1 + \epsilon$. For n_s sufficiently large we must have n_s divisible by either 3^b or 2^d and hence either $2^a 3^{-b} n_s \in S$ or $3^c 2^{-d} n_s \in S$. In either case we see $n_{s+1} \leqslant (1 + \epsilon)n$ so we have $\lim_{s \to \infty} n_{s+1}/n_s = 1$.

Now fix $\epsilon > 0$ and let β *any* real number such that there is an interval (t_0, t_1) for which

$$\psi(t)/t^2 + c/t^2 \leqslant \beta + \epsilon, \qquad t_0 \leqslant t \leqslant t_1. \tag{3.7}$$

By (3.6) we see $\psi(u)/u^2 \leqslant \beta + \epsilon$ for all

$$n \in \bigcup_{j,k} 2^j 3^k (t_0, t_1) = \bigcup_s n_s(t_0, t_1).$$

But $n_s t_1 > n_{s+1} t_0$ for all s such that $n_{s+1}/n_s \leqslant t_1/t_0$, i.e., for all s sufficiently large.

By taking $\beta = \max_{1 \leqslant t \leqslant 2}\{\psi(t)/t^2 + c/t^2\}$ we see that $\limsup \psi(t)/t^2 \leqslant \beta < \infty$. Then by taking $\beta = \liminf \psi(t)/t^2$ we see $\limsup \psi(t)/t^2 \leqslant \liminf \psi(t)/t^2$. ∎

When we apply the preceding lemma to the function $\phi(t)$ we obtain after some simplification the basic asymptotic relation for the Borel transform of ED_n,

$$e^{-\lambda} \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} ED_n \sim \beta\sqrt{\lambda}, \qquad \text{as} \quad \lambda \to \infty. \tag{3.8}$$

4. Tauberian step.

LEMMA 4.1. *The relation*

$$\lim_{\lambda \to \infty} e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} a_n = c \tag{4.1}$$

implies $a_n \to c$ *if and only if*

$$\lim_{\epsilon \to 0^+} \liminf_{n \to \infty} \min_{n \leqslant m \leqslant n + \epsilon\sqrt{n}} \{a_m - a_n\} \geqslant 0. \tag{4.2}$$

REMARKS. This is the Tauberian theorem for Borel summability due to R. Schmidt (1925). For a discussion in a modern probabilistic context and many related references one may consult Bingham (1981).

The preceding lemma does not apply directly to the asymptotic behavior of ED_n, but we will shortly show that the choice $a_n = -n^{-1/2} ED_n$ will complete the proof of Theorem 1.

LEMMA 4.2. (a) $ED_{n+1} \leqslant ED_n + 2\sqrt{2}$,

(b) $\lim_{\lambda \to \infty} e^{-\lambda} \sum_{n=2}^{\infty} (\lambda^n/n!)(ED_n)/\sqrt{n} = c$, $0 < c < \infty$.

PROOF. The first inequality follows from Rédei's algorithm since the passage from D_n to D_{n+1} adds at most two edges which are each bounded by $\sqrt{2}$. One consequence of this inequality is the crude bound

$$ED_n \leqslant 5n. \tag{4.3}$$

Now setting

$$h(\lambda) = e^{-\lambda} \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} (ED_n)/\sqrt{n}$$

we see

$$h(\lambda) \leqslant e^{-\lambda} \sum_{n=0}^{s} \frac{\lambda^n}{n!} (5\sqrt{n}) + e^{-\lambda} s^{-1/2} \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} ED_n, \tag{4.4}$$

so letting $s = [(1-\epsilon)\lambda]$ we see from (4.4) and (3.8)

$$\limsup_{\lambda\to\infty} h(\lambda) \leqslant (1-\epsilon)^{-1/2}c, \qquad \text{for all} \quad \epsilon > 0. \tag{4.5}$$

The lower bound is just as easy since for $S = [(1+\epsilon)\lambda]$,

$$h(\lambda) \geqslant e^{-\lambda}s^{-1/2}\sum_{n=2}^{s}\frac{\lambda^n}{n!}ED_n \geqslant e^{-\lambda}s^{-1/2}\left(\phi(\lambda) - \sum_{n=s+1}^{\infty}\frac{\lambda^n}{n!}5\sqrt{n}\right). \tag{4.6}$$

From (3.8) and (4.6) we have

$$\liminf_{\lambda\to\infty} h(\lambda) \geqslant (1+\epsilon)^{-1/2}c, \tag{4.7}$$

and the inequalities (4.5) and (4.7) naturally complete the proof of the lemma. ∎

LEMMA 4.3. *For $a_n = -n^{-1/2}ED_n$ we have*

$$\lim_{\epsilon\to 0^+}\liminf_{n\to\infty}\min_{n<m\leqslant n+\epsilon\sqrt{n}}\{a_m - a_n\} \geqslant 0. \tag{4.8}$$

PROOF. We have to bound

$$\min_{n<m\leqslant n+\epsilon\sqrt{n}}\{a_m - a_n\} = -\max_{n<m\leqslant n+\epsilon\sqrt{n}}\{m^{1/2}ED_m - n^{1/2}ED_n\}.$$

To do so we note $xy - x'y' = (x-x')y + x'(y-y')$, so for $n \leqslant m \leqslant n+\epsilon\sqrt{n}$

$$\begin{aligned} m^{-1/2}ED_m &\leqslant n^{-1/2}ED_n + \sum_{k=1}^{m-n}\left\{(n+k)^{-1/2}ED_{n+k} - (n+k-1)^{-1/2}ED_{n+k-1}\right\} \\ &\leqslant n^{-1/2}ED_n + \sum_{k=1}^{m-n}(n+k-1)^{-1/2}E(D_{n+k} - D_{n+k-1}) \\ &\leqslant n^{-1/2}ED_n + \epsilon 2\sqrt{2}\,. \end{aligned} \tag{4.9}$$

One then sees that for all n,

$$\min_{n<m\leqslant n+\epsilon\sqrt{n}}\{a_m - a_n\} \geqslant -\epsilon 2\sqrt{2} \tag{4.10}$$

so (4.8) follows immediately. ∎

With the assumption of the last two lemmas we are able to conclude that $ED_n \sim \beta\sqrt{n}$ as $n\to\infty$ and thus complete the proof of Theorem 1.

5. An efficient algorithm. The algorithm given here is based on geometric partitioning and dynamic programming. We first recall that the m city TSP can be solved by a dynamic programming in time $O(m2^m)$ (Bellman 1960, Held and Karp 1972). Almost without modification the dynamic programming algorithm can be used on the DTSP and the same time bound holds.

Now we spell out the DTSP analogue to Algorithm A of Karp (1977). First we choose a real sequence $t(n)$ satisfying

$$\log\log_2 n \leqslant t(n) \leqslant 4\log\log_2 n, \tag{5.1}$$

$$(n/t(n))^{1/2} = 2^j \qquad \text{for some} \quad j = 0, 1, \ldots\,. \tag{5.2}$$

We also need some notation for a decomposition of the unit square $Q = [0,1]^2$. We let Q_i, $1 \leqslant i \leqslant 4$, denote its four quadrants, and for each i we let Q_{ij}, $1 \leqslant j \leqslant 4$ denote the four quadrants of Q_i. More generally, for each s we let $Q_{i_1 i_2 \dots i_s}$, $1 \leqslant i_s \leqslant 4$, denote the four quadrants of $Q_{i_1 i_2 \dots i_{s-1}}$. We can now sketch the procedure.

DTSP Algorithm. (1) Decompose $[0,1]^2$ into 4^k subsquares $Q_{i_1 i_2 \dots i_k}$ for $k = \frac{1}{2}\log_2(n/t(n))$.

(2) Use dynamic programming to find an optimal solution for the DTSP in each subsquare $Q_{i_1 i_2 \dots i_k}$.

(3) For $s = k$ until 1 and for all $1 \leqslant i_j \leqslant r$, $1 \leqslant j \leqslant s$, sew the paths obtained in $Q_{i_1 i_2 \dots i_s}$, $1 \leqslant i_s \leqslant 4$ together by the method of §3 in order to get a path through $Q_{i_1 i_2 \dots i_{s-1}}$. ∎

It is easy to see that this algorithm runs in expected time $O(n \log n)$. It remains to check that under the model of the random DTSP studied here that the algorithm provides a nearly optimal solution.

If we let $O_{i_1 i_2 \dots i_k}$ denote the length of the optimal path through the vertices of $Q_{i_1 i_2 \dots i_k}$ then inequality (2.9) says that these can be sewed together to get a path through the vertices of $Q_{i_1 i_2 \dots i_{k-1}}$ of length $L_{i_1 i_2 \dots i_{k-1}}$ satisfying

$$L_{i_1 i_2 \dots i_{k-1}} \leqslant \sum_{1 \leqslant i_s \leqslant 4} O_{i_1 i_2 \dots i_k} + c 2^{-k}. \tag{5.3}$$

If $L_{i_1 i_2 \dots i_s}$ is the length of the path through the vertices of $Q_{i_1 i_2 \dots i_s}$, $1 \leqslant s \leqslant k$, obtained by our algorithm then using inequality (2.9) starting from (5.3) we find for $s = k - j$

$$L_{i_1 i_2 \dots i_{k-j}} \leqslant \sum_{\substack{1 \leqslant i_t \leqslant 4 \\ k-j+1 \leqslant t \leqslant k}} O_{i_1 i_2 \dots i_k} + c \sum_{t=1}^{j} 2^{-k+j-t} \cdot 4^{t-1}. \tag{5.4}$$

If we let D_n^* denote the length of the path given by the algorithm, then setting $j = k$ in (5.4) and simplifying will give

$$D_n^* \leqslant \sum_{\substack{1 \leqslant i_j \leqslant 4 \\ 1 \leqslant j \leqslant k}} O_{i_1 i_2 \dots i_k} + c 2^k. \tag{5.5}$$

Now since $Q_{i_1 i_2 \dots i_k}$ has area $p = 4^{-k} = t(n)/n$ the number of vertices in $Q_{i_1 i_2 \dots i_k}$ is distributed as a binomial random variable with parameters n and p we see by conditioning that

$$EO_{i_1 i_2 \dots i_k} = \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} E(D_j) \cdot 2^{-k}. \tag{5.6}$$

From (5.5) we get

$$ED_n^* \leqslant \sqrt{n/t(n)} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} E(D_j) + c\sqrt{n/t(n)}\,. \tag{5.7}$$

Since $pn = t(n)$ and since $ED_k \sim \beta\sqrt{k}$ the binomial sum in (5.7) is asymptotic to $\beta\sqrt{t(n)}$. From this we see the whole right side of (5.7) is asymptotic to $\beta\sqrt{n}$. This bound completes the proof of Theorem 2.

6. Open problem. The results of this article are pointed toward the establishment of algorithms which perform well in terms of the *expected* length of the solution obtained. This is somewhat in contrast to the original conception of Karp (1977) where a similar algorithm is shown to provide an ϵ-optimal solution with probability one (see also Steele 1981b and Weide 1978 for some subsequent refinements and elaboration of Karp's theory.)

It is natural to ask if the present model will also yield an ϵ-optimal algorithm with probability one. The basic step would consist in proving the natural conjecture

$$D_n \sim \beta\sqrt{n}\,, \qquad \text{with probability one.} \tag{6.1}$$

There are three approaches which have succeeded in similar problems (Beardwood, Halton and Hammersley 1959 and Steele 1981a, c, but these methods do not seem capable of proving (5.1).

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