RANDOM WALKS WHOSE CONCAVE MAJORANTS OFTEN HAVE FEW FACES

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ABSTRACT. We construct a continuous distribution G such that the number of faces in the smallest concave majorant of the random walk with G-distributed summands will take on each natural number infinitely often with probability one. This investigation is motivated by the fact that the number of faces F_n of the concave majorant of the random walk at time n has the same distribution as the number of records R_n in the sequence of summands up to time n. Since R_n is almost surely asymptotic to $\log n$, the construction shows that despite the equality of all of the one-dimensional marginals, the almost sure behaviors of the sequences $\{R_n\}$ and $\{F_n\}$ may be radically different.

1. Introduction

If X_i, $i = 1, 2, \ldots$ is a sequence of independent random variables with a continuous distribution G, then the number of records

$$R_n = \max\{k : X_{i_1} < X_{i_2} < \ldots < X_{i_k}, 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

was studied in Rényi (1962) and was found to have the same distribution as

$$\xi_1 + \xi_2 + \cdots + \xi_n \tag{1}$$

where $\{\xi_i : i = 1, 2, \ldots\}$ is a sequence of independent Bernoulli random variable that satisfy $P(\xi_k = 1) = 1/k$. Goldie (1989) later observed that as a consequence of Spitzer's combinatorial lemma as generalized by Brunk (1964) that the number of faces of the concave majorant of the random walk $S_k = X_1 + X_2 + \cdots + X_k$, $1 \leq k \leq n$, also has the same distribution as R_n; that is, if one lets F_n denote the number of pieces in the smallest piecewise linear concave majorant of the set of points $\mathcal{S}_n = \{ (0,0), (1, S_1), \ldots, (n, S_n) \}$, then one has $P(R_n \leq t) = P(F_n \leq t)$ for all $t \in \mathbb{R}$ and all integers $1 \leq n < \infty$.

By a standard Borel-Cantelli argument, one finds from the Bernoulli sum representation (1) of R_n and the monotonicity $R_n \leq R_{n+1}$ that

$$\lim_{n\to\infty} R_n / \log n = 1 \quad \text{with probability 1,} \tag{2}$$

so from Goldie's observation that $R_n \stackrel{d}{=} F_n$ for each n, one might expect an analogous strong law for the sequence $\{F_n : n = 1, 2, \ldots\}$, despite the fact that the sequence $\{F_n : n = 1, 2, ...\}$ is not monotone. In Steele (2002) it was suggested the

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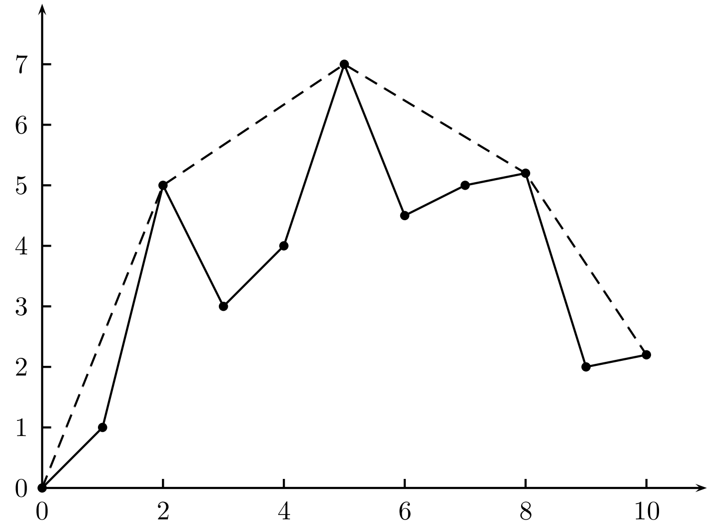

Figure 1. The concave majorant of a random walk

limit law for records (2) might *not* extend to the face process $\{F_n : n = 1, 2, ...\}$, and the main goal of this note is to confirm a particularly strong version of this conjecture.

Theorem 1. *There exists a continuous distribution function G such that if the random variables X_i, $i = 1, 2, \ldots$ are independent with $P(X_i \le x) = G(x)$ for all $i = 1, 2, ...$ and if F_n denotes the number of faces of the least concave majorant of the first n steps of the random walk S_k, $1 \le k \le n$, then*

$$P(F_n = m \quad i.o.) = 1 \quad \text{for each value of } m = 1, 2, ... \tag{3}$$

In particular, the concave majorant of the set $\mathcal{S}_n \subset \mathbb{R}^2$ will be a single line infinitely often with probability one.

The behavior described by the relation (3) contrasts about as sharply with the limit (2) as one could imagine, despite the fact that the marginal distributions of F_n and R_n are equal for each n.

2. Two Constructive Lemmas

The basic idea is that one can construct a continuous distribution G such that infinitely often the summand X_i is so large that it completely overwhelms the cumulative contributions of all of the preceding summands. The implementation of this idea rests on two simple technical lemmas. To begin, we take an arbitrary sequence of integers $2 \le n_1 < n_2 < \cdots$, and consider independent *discrete* random variables Y_i, $i = 1, 2, \ldots$ such that $P(Y_i = n_k) = p_k$ for all $i \ge 1$ and $k \ge 1$. Our technical lemmas tell us about events that are unlikely to take place in conjunction with the first time that an element of the sequence $\{Y_i\}$ is equal to n_k.

Lemma 1. *For $\alpha > 1$, let c_α be chosen such that $p_k = c_\alpha(k!)^{-\alpha}$ is a probability measure on $\{1, 2, ...\}$. If one sets*

$$B_{k,t} = \{Y_t = n_k\} \cap \{Y_s \neq n_k \text{ for all } 1 \leq s < t\} \cap \{\max_{s<t} Y_s > n_k\},$$

then for $A_k = \bigcup_{t=1}^{\infty} B_{k,t}$ one has $P(A_k \quad i.o.) = 0$.

Proof. If we set $\alpha_k = \sum_{j=1}^{k-1} p_j/(1-p_k)$, then by independence of the $\{Y_k\}$ we have

$$P(B_{k,t}) = p_k(1-p_k)^{t-1}(1-\alpha_k^{t-1}).$$

Since the events $\{B_{k,t}\}_{t=1}^{\infty}$ are disjoint, we also have

$$\sum_{k=1}^{\infty} P(A_k) = \sum_{k=1}^{\infty}\sum_{t=1}^{\infty} P(B_{k,t}) = \sum_{k=1}^{\infty} p_k\Big(\frac{1}{p_k} - \frac{1}{1-(1-p_k)\alpha_k}\Big) \tag{4}$$

$$= \sum_{k=1}^{\infty} \frac{1-\sum_{j=1}^{k} p_j}{1-\sum_{j=1}^{k-1} p_j} = \sum_{k=1}^{\infty} \frac{\sum_{j=k+1}^{\infty} p_j}{\sum_{j=k}^{\infty} p_j} \tag{5}$$

$$\leq \sum_{k=1}^{\infty} \frac{1}{(k+1)^\alpha} < \infty, \tag{6}$$

and the proof is completed by applying the Borel-Cantelli Lemma. □

Remark: One should note that with the choice $p_k = c_\alpha(k!)^{-\alpha}$ the condition $\alpha > 1$ cannot be dropped. For example, one can easily check that if $p_k = e/k!$, then the sum of the $P(A_k)$ diverges. On the other hand, a referee has observed that an interesting alternative that does work here (and in the next lemma) is given by $p_k = q^{k^2}$ for any $0 < q < 1$. With this choice the inequalities (4) and (5) are unchanged but the last line is somewhat simplified. Specifically, with $p_k = q^{k^2}$ the bound in (6) can be replaced by $q^3 + q^5 + q^7 + \cdots < \infty$.

Lemma 2. *Suppose the sequence $\{n_k\}$ satisfies the gap condition $n_k/n_{k-1} > k/p_k$ for all $k \geq 2$. If we have*

$$E_{k,t} = \{Y_t = n_k\} \cap \{Y_s < n_k, \text{ for all } 1 \leq s < t\} \quad \text{and} \quad E_k = \cup_{t \geq n_k/n_{k-1}} E_{k,t},$$

then we have $P(E_k \text{ i.o.}) = 0$.

Proof. We have $P(E_{k,t}) = p_k(\sum_{i=1}^{k-1} p_i)^{t-1}$, so by disjointness we find

$$\sum_{k=1}^{\infty} P(E_k) = \sum_{k=1}^{\infty} \sum_{t=\lceil \frac{n_k}{n_{k-1}} \rceil}^{\infty} p_k\Big(\sum_{i=1}^{k-1} p_i\Big)^{t-1} = \sum_{k=1}^{\infty} p_k \frac{\big(\sum_{i=1}^{k-1} p_i\big)^{\lceil \frac{n_k}{n_{k-1}} \rceil - 1}}{1-\sum_{i=1}^{k-1} p_i}$$

$$\leq \sum_{k=1}^{\infty} p_k \frac{(1-p_k)^{\lceil \frac{n_k}{n_{k-1}} \rceil - 1}}{p_k} \leq \sum_{k=1}^{\infty} (1-p_k)^{(k/p_k)-1}.$$

From the bound $1 - x \leq e^{-x}$ and the geometric sum, one sees the last sum is not larger than $(1-e^{-1})^{-1}$, so the Borel-Cantelli Lemma again completes the proof. □

3. Proof of Theorem 1

We now say that a random time τ is *good* provided that

- $\tau \geq 2$,
- $Y_j < Y_\tau$ for all $1 \leq j < \tau$, and
- for n_k such that $Y_\tau = n_k$ one has $\tau < n_k/n_{k-1}$,

so, the main point of Lemmas 1 and 2 is that they immediately imply that with probability one there exists an infinite sequence $\tau_1 < \tau_2 < ...$ of good times. To complete the proof of Theorem 1 we just need to connect the existence of these good times to the geometry of the concave majorant of an appropriate random walk.

Specifically, we let U_n, $n = 1, 2, ...$ denote a sequence of independent random variables with the uniform distribution on $(0, 1)$, and we set $X_n = Y_n + U_n$, where the Y_n are as before. We will now focus on the random walk $S_n = X_1 + X_2 + \cdots + X_n$ and confirm that the continuous distribution $G(x) = P(X_k \leq x)$ satisfies the claims of Theorem 1.

The key geometric fact of the construction, fits snugly into a single line:

$$\text{if } \tau \text{ is a good time, then } F_\tau = 1. \tag{7}$$

To see why this assertion is true, we first note that

$$\frac{S_\tau}{\tau} > \frac{n_k + (\tau - 1)n_1}{\tau} > n_{k-1} + n_1 - n_1/\tau \geq n_{k-1} + 1, \tag{8}$$

where in the last step we use the facts that $n_1 \geq 2$ and $\tau \geq n_1$. We then note that for all $t < \tau$ we have

$$\frac{S_t}{t} < \frac{t(n_{k-1} + 1)}{t} = n_{k-1} + 1, \tag{9}$$

and the truth of the assertion (7) follows immediately from the bounds (8) and (9).

By assertion (7) and the almost sure existence of an infinite sequence of good times, we therefore find that $F_n = 1$ infinitely often with probability one. Now we only need to check that for each $m \geq 1$ we also have $F_n = m$ infinitely often with probability one. The basic idea here is that we get infinitely many independent tries at an event that has probability that is uniformly bounded away from zero.

More formally, since the summands $\{X_k = Y_k + U_k : k = 1, 2, ...\}$ are nonnegative, elementary geometry tells us that for each good time τ and for each $m \geq 2$ that the event

$$S_\tau/\tau > X_{\tau+1} > X_{\tau+2} > \cdots > X_{\tau+m-1}$$

implies the event $F_{\tau+m-1} = m$. Also, the event $X_{\tau+1} > X_{\tau+2} > \cdots > X_{\tau+m-1}$ has probability greater than that of the event

$$C_m = \{Y_{\tau+1} = Y_{\tau+2} = \cdots = Y_{\tau+m-1} = n_1\} \cap \{U_{\tau+1} > U_{\tau+2} > \cdots > U_{\tau+m-1}\},$$

and C_m has probability $p_1^{m-1}/(m-1)! = \delta_m > 0$, which does not depend on τ. Moreover, by (8) one always has $S_\tau/\tau > n_1 + 1$ provided that $Y_\tau \geq n_2$, so along the infinite sequence of good times τ_j, $j = 2, 3, ...$ one can find has infinitely many opportunities of observing $F_{\tau+m-1} = m$ that are independent and that have probability $\delta_m > 0$. Thus, by the law of large numbers one finds that with probability one, we have $F_n = m$ for infinitely many n, and the proof of Theorem 1 is complete.

4. A Final Observation

The distribution G constructed here suffices to show that one cannot expect regular behavior of F_n at the same level of generality that one finds regularity for R_n. Nevertheless, the sequence F_n may not always be badly behaved. Under nice conditions — say, for example, when the summands are exponentially distributed — one may be able to prove a useful limit law for F_n.

In the hunt for such a law, it may be useful to note that for summands with a continuous distribution one always has

$$\limsup_{n\to\infty} \frac{F_n}{\log n} \geq 1 \quad \text{with probability one.} \tag{10}$$

Moreover, from the equality of the distributions of R_n and F_n and the representation (1) for R_n, one has the useful large deviation bound:

$$P(|F_n - H_n| \geq \epsilon H_n) \leq 2\exp(-\epsilon^2 H_n/4) \quad \text{for all} \quad 0 \leq \epsilon \leq 1/2, \tag{11}$$

where $H_n = 1 + 1/2 + 1/3 + \cdots + 1/n$. In fact, the bound (11) follows immediately from the usual concentration inequalities for Bernoulli sums, say, for example, Bennet's inequality (Bennett (1962), equation (8b)). Finally, from (11) one easily proves (10) with the Borel-Cantelli lemma and a subsequence argument.

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