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Sharper Wiman Inequality for Entire Functions with Rapidly Oscillating Coefficients

J. MICHAEL STEELE*

Department of Statistics, E-220 Engineering Quadrangle, Princeton University, Princeton, New Jersey 08544

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For entire functions of the form $\sum_{n=0}^{\infty} a_n e^{i\theta_n t} z^n$ where the θ_n are integers satisfying the Hadamard gap condition it is proved that Wiman's inequality can be improved to $M(r) \leqslant \mu(r)\ (\log \mu(r))^{1/4}\ (\log\log \mu(r))^{1+\delta}$, for almost every t and all r except a set $E_\delta(t)$ of finite logarithmic measure. © 1987 Academic Press, Inc.

I. INTRODUCTION

Wiman [9] proved that any entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with maximum modulus $M(r) = \max_{|z|=r} |f(z)|$ and maximal term $\mu(r) = \max_n |a_n|\, r^n$ satisfies

$$M(r) \leqslant \mu(r)(\log \mu(r))^{1/2+\delta}, \tag{1.1}$$

for all $\delta > 0$ and all $0 < r < \infty$ except a set E_δ of finite logarithmic measure $(\int_{E_\delta} (dr/r) < \infty)$.

In Rosenbloom [6] an elegant probabilistic method was introduced which used the theory of exponential families and Chebyshev's inequality to obtain a sharper form of Wiman's theorem

$$M(r) \leqslant \mu(r)(\log \mu(r))^{1/2}\ (\log\log \mu(r))^{1+\delta} \tag{1.2}$$

for all $\delta > 0$ and all $0 < r < \infty$ except a set E_δ of finite logarithmic measure. Just by considering e^z, one can see that (1.2) is the best possible result as far as the exponent of $\log \mu(r)$ is concerned.

This is in contrast to the result of Lévy [5] on random entire functions,

$$f(z, \omega) = \sum_{n=0}^{\infty} a_n e^{i\theta_n(\omega)} z^n, \tag{1.3}$$

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where $\theta_n(\omega)$ are independent uniformly distributed random variables on $[0, 2\pi]$. Under a regularity condition on $f(z)$ (satisfied when $a_n = 1/n!$), Lévy was able to show that with probability one,

$$M(r, \omega) \leqslant \mu(r) (\log \mu(r))^{1/4 + \delta} \tag{1.4}$$

for all $\delta > 0$ and all $0 < r < \infty$ except a set $E_\delta(\omega)$ of finite logarithmic measure.

Building on the technique of Rosenbloom, Erdös, and Rényi [1] (1969) were able to remove the regularity restrictions in Lévy's theorem and to obtain for (1.3) the sharper result that with probability one,

$$M(r, \omega) \leqslant \mu(r) (\log \mu(r))^{1/4} (\log \log \mu(r))^{1 + \delta} \tag{1.5}$$

for all $\delta > 0$ and all $0 < r < \infty$ except a set $E_\delta(\omega)$ of finite logarithmic measure.

The objective of this article is to establish the analog of the Erdös–Rényi theorem for the class of entire functions

$$f(z, t) = \sum_{n=0}^{\infty} a_n e^{it\theta_n} z^n \tag{1.6}$$

where θ_n is a fixed sequence of integers satisfying the Hadamard gap condition $\theta_{n+1}/\theta_n \geqslant q > 1$ for $n \geqslant 0$. Explicitly, we have

THEOREM 1. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is entire and θ_n are positive integers satisfying $\theta_{n+1}/\theta_n \geqslant q > 1$, then for any $\delta > 0$ and for almost every t the maximum modulus of* (1.6), *$M(r, t) = \max_{|z| = r} |f(z, t)|$, satisfies*

$$M(r, t) < \mu(r) (\log \mu(r))^{1/4} (\log \log \mu(r))^{1 + \delta} \tag{1.7}$$

for all $0 < r < \infty$ except a set $E_\delta(t)$ of finite logarithmic measure.

This result will be obtained as a corollary to a slightly stronger one which is also analogous to a theorem of Erdös and Rényi [1]. This time the bound on $M(r)$ is expressed in terms of $S(r)$, where $S^2(r) = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$.

THEOREM 2. *Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is entire and θ_n are integers satisfying $\theta_{n+1}/\theta_n \geqslant q > 1$ for all $n \in \mathbb{Z}^+$. There exists a constant c (not depending on r or t) such that for almost every t,*

$$M(r, t) \leqslant cS(r) (\log \log \mu(r))^{1/2} \tag{1.8}$$

for all $0 < r < \infty$ except a set $E_\delta(t)$ of finite logarithmic measure.

To see that Theorem 2 implies Theorem 1 we note that Rosenbloom's inequality (1.5) applied to $f_1(z)=\sum_{n=0}^{\infty}|a_n|\,z^n$ implies

$$S^2(r)\leqslant \mu(r)\sum_{n=0}^{\infty}|a_n|\,r^n\leqslant \mu^2(r)\,(\log \mu(r))^{1/2}\,(\log\log \mu(r))^{1+\delta} \tag{1.9}$$

for all $r\notin E_\delta$. By (1.8), we then have

$$M(r,t)\leqslant c\mu(r)\,(\log\ \mu(r))^{1/4}\,(\log\log \mu(r))^{1+\delta/2} \tag{1.10}$$

for all $r\notin E_\delta\cup E_\delta(t)\equiv E'_\delta(t)$, and this inequality is equivalent to (1.7).

It therefore remains only to prove Theorem 2. For this there are basically two steps. The first of these is the derivation of a maximal inequality, and the second is an interpolation argument. These are carried out in Sections 2 and 3. In Section 4 a probabilistic argument is used to show that Theorem 3 is essentially best possible.

As a last introductory point, one should note that the results of the present paper are more in the domain of Wiman's inequality and it's extensions rather than in the explicit domain of random series as typified by the work of Kahane [4] which would now be considered the standard reference in the theory of random series.

Two relevant works, which were not covered in Kahane [4], are Takafumi [7, 8] where progress is reported on the behavior of random series with gaps. A recent work on the non-probabilistic growth aspects of analytic functions is Juneja and Kapoor [3]. That work develops a sharper form of Wiman's inequality than that given by Rosenbloom [6], and it provides exercises which explore the senses in which their sharper inequality cannot be improved. Juneja and Kapoor [3] do not consider random series and they do not pursue the work begun in Erdös and Rényi [1]. The sharpest inequalities in the non-random case are of necessity less sharp than the Erdös–Rényi inequality (1.5).

II. Maximal Inequality

For brevity and to stress the parellel with the probabilistic methods of Erdös and Rényi [1], we write $P(A)=(2\pi)^{-1}\int_A dt$ for any Borel $A\subset[0,2\pi]$. For any function $f(t)$, $A=\{f(t)>\lambda\}$ will be used to abbreviate $A=\{0\leqslant t\leqslant 2\pi\colon f(t)>\lambda\}$. Before proving our maximal inequality we establish a result of large deviation type for lacunary polynomials. The technique used depends on an elegant representational device due to Jakubowski and Kwapién [2].

LEMME 2.1. *For any $q>1$ there are constants A_q and B_q (depending only on q) such that any positive integers n_k, $1 \leqslant k \leqslant N$, which satisfy $n_{k+1}/n_k \geqslant q$, we have for any complex numbers a_k,*

$$P\left(\left|\sum_{k=1}^{N} a_k(\cos n_k t)\right| > A_q \lambda \left(\sum_{k=1}^{N} |a_k|^2\right)^{1/2}\right) \leqslant B_q e^{-\lambda^2}. \tag{2.1}$$

Proof. To begin with, we will suppose that the a_k are real and that $q \geqslant 2$. In this case, we note that the sum of the elements of each of the 2^N subsets of $\{n_1, n_2, ..., n_N\}$ are distinct. For each $1 \leqslant k \leqslant N$ we let $r_k(\omega)$ denote the kth Rademacher function, i.e., $r_k(\omega) = \text{sign}(\sin 2^k \pi \omega)$ for $0 \leqslant \omega \leqslant 1$. Now we let

$$f(\omega, t) = \prod_{k=1}^{N} (1 + r_k(\omega) \cos n_k t) \tag{2.2}$$

and note that $f(\omega, t)$ is a probability density with respect to the product measure $d\omega dP$, i.e., $f(\omega, t) \geqslant 0$ and $\int_0^{2\pi} \int_0^1 f(\omega, t)\, d\omega\, dP = 1$.

The point of introducing (2.2) is the representation

$$\cos n_k t = \int_0^1 r_k(\omega) f(\omega, t)\, d\omega \tag{2.3}$$

which yields the basic identity

$$\sum_{k=1}^{N} a_k \cos n_k t = \int_0^1 \left(\sum_{k=1}^{N} a_k r_k(\omega)\right) f(\omega, t)\, d\omega. \tag{2.4}$$

By Markov's inequality, (2.4), and Jensen's inequality (respectively) we have for all $\alpha > 0$,

$$\begin{aligned} P\left(\left|\sum_{k=1}^{N} a_k \cos n_k t\right| \geqslant \lambda\right) &\leqslant e^{-\alpha\lambda} \int_0^{2\pi} \exp\left(\alpha \left|\sum_{k=1}^{N} a_k \cos n_k t\right|\right) \frac{dt}{2\pi} \\ &\leqslant e^{-\alpha\lambda} \int_0^{2\pi} \int_0^1 \exp\left(\alpha \left|\sum_{k=1}^{N} a_k r_k(\omega)\right|\right) f(\omega, t)\, d\omega \frac{dt}{2\pi} \\ &= e^{-\alpha\lambda} \int_0^1 \exp\left(\left|\sum_{k=1}^{N} a_k r_k(\omega)\right|\right) d\omega. \end{aligned} \tag{2.5}$$

The Rademacher functions are independent when viewed as random variables so recalling $e^{-x} + e^x \leqslant 2e^{x^2}$ and setting $S^2 = \sum_{k=1}^{N} a_k^2$ one has

$$\int_0^1 \exp\left(\alpha \sum_{k=1}^{N} a_k r_k(\omega)\right) d\omega = \prod_{k=1}^{N} \frac{1}{2} (e^{\alpha a_k} + e^{-\alpha a_k}) \leqslant e^{\alpha^2 S^2}. \tag{2.6}$$

Since also $e^{|x|} \leqslant e^{-x} + e^{x}$, (2.5) and (2.6) yield

$$P\left(\left|\sum_{k=1}^{N} a_k \cos n_k t\right| \geqslant \lambda\right) \leqslant 2 \exp(\alpha^2 S^2 - \alpha\lambda), \tag{2.7}$$

so choosing α to minimize the exponent yields an upper bound of $2\exp(-\lambda^2/4S^2)$.

This completes the proof of the lemma in case of real a_k and $q \geqslant 2$. In fact it shows $A_q = 2$ and $B_q = 2$ suffice for that case. For a_k still real but for $1 < q \leqslant 2$, we need to choose a new integer s such that $q^s \geqslant 2$, e.g., $s = [(\log_2 q)^{-1}] + 1$. We now write

$$\sum_{k=1}^{N} a_k \cos n_k t = \sum_{j=0}^{s-1} \sum_{k \equiv j \bmod s} a_k \cos n_k t, \tag{2.8}$$

and note by our result for $q \geqslant 2$ that by (2.7) and Schwarz's inequality

$$\begin{aligned} &P\left(\left|\sum_{k=1}^{N} a_k \cos n_k t\right|^2 \geqslant \lambda^2 \sum_{k=1}^{N} a_k^2\right) \\ &\quad\leqslant \sum_{j=0}^{s-1} P\left(s\left|\sum_{k \equiv j \bmod s} a_k \cos n_k t\right|^2 \geqslant \lambda^2 \sum_{k \equiv j \bmod s} a_k^2\right) \\ &\quad\leqslant 2s \exp(-\lambda^2/4s). \end{aligned}$$

This completes the proof of the lemma for real a_k and specifies the values of A_q and B_q. The inequality $|u + iv| \leqslant \sqrt{2}\max(|x|, |y|)$ shows that the real case of (2.1) suffices to imply the complex case. ∎

We can now prove our basic maximal inequality.

LEMMA 2.2. *For any $q > 1$ there is a constant $c_{q,\beta}$ (depending only on q and β) such that we have the following:*

For all complex numbers c_k, $1 \leqslant k \leqslant N$, and all positive integers n_k satisfying $n_{k+1}/n_k \geqslant q$ we have

$$P\left(\max_{0 \leqslant \phi \leqslant 2\pi}\left|\sum_{k=1}^{N} c_k e^{ik\phi} \cos n_k t\right| \geqslant c_{q,\beta} S(\log N)^{1/2}\right) \leqslant N^{-\beta} \tag{2.10}$$

where $S^2 = \sum_{k=1}^{N} |c_k|^2$.

Proof. We let M be a positive integer and set $\phi_j = 2\pi j/M$ for $j = 0, 1, \ldots, M-1$. By Schwarz's inequality and the fact that $|e^{ia} - e^{ib}| \leqslant |a - b|$ for real a, b; we have

$$\max_{0 \leqslant \phi \leqslant 2\pi}\left|\sum_{k=1}^{N} c_k e^{ik\phi} \cos n_k t\right| \leqslant \max_{0 \leqslant j < M}\left|\sum_{k=1}^{N} c_k e^{ik\phi_j} \cos n_k t\right| + 2\pi S M^{-1} N^{3/2}. \tag{2.11}$$

Choosing $M=[4\pi N^{3/2}A_q^{-1}\lambda]+1$ then setting $\lambda=2\sqrt{\alpha\log N}$ (for $N\geqslant 3$), one can simplify the inequalities

$$P\left(\max_{0\leqslant\phi\leqslant 2\pi}\left|\sum_{k=1}^{N} c_k e^{ik\phi}\cos n_k t\right|\geqslant A_q S\lambda\right)$$
$$\leqslant\sum_{j=0}^{M-1} P\left(\left|\sum_{k=1}^{N} c_k e^{ik\phi_j}\cos n_k t\right|\geqslant A_q S\lambda-2\pi SM^{-1}N^{3/2}\right)\leqslant MB_q e^{-\lambda^2/4} \tag{2.12}$$

in order to obtain (2.10) for $0\leqslant\beta<\alpha-\frac{3}{2}$. Since $\alpha>0$ is arbitrary the proof is complete. ∎

Remark. The preceding result for $\cos n_k t$ is just as valid for $\sin n_k t$, so in (2.10) we can replace $\cos n_k t$ by $\exp(in_k t)$ at the expense of replacing $c_{q,\beta}$ by $2c_{q,\beta}$.

III. Proof of Theorem 2

First we give an interpolation argument which parallels Erdös and Rényi [1] although we deal directly with S instead of μ.

For a given entire $f(z)=\sum_{n=0}^{\infty} a_n z_n$ we set $f_1(z)=\sum_{n=0}^{\infty}|a_n|\,z^n$ and let E_δ denote the exceptional set in Rosenbloom's inequality (1.2) as applied to $f_1(z)$. Without loss of generality we can assume E_δ is the union of disjoint open intervals (a_j, b_j), $1\leqslant j<\infty$. Next, we set $S^2(r)=\sum_{n=0}^{\infty}|a_n|^2 r^{2n}$ and proceed to define recursively a sequence $\{r_k\}$ of interpolation nodes for $S(r)$.

Given r_k we let $r_k^*=\inf\{r\colon \log S(r)\geqslant 1+\log S(r_k)\}$. If $r_k^*\notin E_\delta$ we let $r_{k+1}=r_k^*$, but if $r_k^*\in I_j=(a_j, b_j)$ for some j we let $r_{k+1}=a_j$ and $r_{k+2}=b_j$. To begin the construction we can just take $r_0=0$.

The main properties of $\{r_k\}$ are

$$r_k\notin E_\delta,\qquad \text{for all } 0<k<\infty, \tag{3.1}$$

$$\log S(r_k)\geqslant[k/2], \tag{3.2}$$

$$\log S(r_{k+1})=1+\log S(r_k)\qquad \text{if the interval } (r_k, r_{k+1}) \text{ contains any point } \textit{not} \text{ in } E_\delta. \tag{3.3}$$

We will first check that to prove Theorem 2 it suffices to show that there is a constant c_0 such that for almost every $t\in[0, 2\pi]$ there is *a* $N(t)$ such that

$$M(r_k, t)<c_0 S(r_k)(\log\log\mu(r_k))^{1/2} \tag{3.4}$$

for all $k\geqslant N(t)$.

To see why (3.4) implies Theorem 2 take $r \in (r_k, r_{k+1}) \cap E_\delta^c$, and note by monotonicity and Rosenbloom's inequality that

$$\begin{aligned} M(r, t) &< c_0 S(r_{k+1})(\log\log \mu(r_{k+1}))^{1/2} \\ &< c_0 e S(r_k)(\log\log S(r_{k+1}))^{1/2} \\ &< c_0 e S(r_k)\{\log(1+\log S(r_k))\}^{1/2} \\ &< c_0 e S(r_k)\{\log(1+\tfrac{1}{2}\log \mu(r_k) + \log f_1(r_k))\}^{1/2} \\ &< c_0 e S(r_k)\{\log(1+\log \mu(r_k) + \tfrac{1}{4}\log\log \mu(r) \\ &\quad + (1+\delta)\log\log\log \mu(r))\}^{1/2}. \end{aligned} \tag{3.5}$$

The last term in (3.5) is clearly majorized by $c_1 S(r)(\log\log \mu(r))^{1/2}$ for $c_1 = c_0 e^2$ and all $r \geqslant R$, where R does not depend on t (only on μ). This completes the argument that Theorem 2 follows from (3.4).

To establish (3.4) we first note that for any $A(r)$ and $C(r)$ that

$$M(r, t) \leqslant \sum_{|n - A(r)| \geqslant C(r)} |a_n|\, r^n + \max_{0 \leqslant \phi \leqslant 2\pi} \left| \sum_{|n - A(r)| \leqslant C(r)} a_n e^{in\phi} r^n \exp(i\theta_n t) \right|. \tag{3.6}$$

We consider a random variable X with $P(X = n) = |a_n|\, r^n / f_1(r)$ and take $A(r) = E(X)$ and $C(r) = TB(r)$, where $B^2(r) = \operatorname{Var} X$. Then by the Chebyshev and Rosenbloom inequalities one has

$$\begin{aligned} \sum_{|n - A(r)| \geqslant TB(r)} |a_n|\, r^n &\leqslant f_1(r)/T^2 \\ &\leqslant \mu(r)(\log \mu(r))^{1/4}(\log\log \mu(r))^{1+\delta}/T^2 \end{aligned} \tag{3.7}$$

for all $r \notin E_\delta$. Letting $T^2 = \log \mu(r)$ we see

$$\sum_{|n - A(r)| > TB(r)} |a_n|\, r^n \leqslant \mu(r) \leqslant S(r) \tag{3.8}$$

for all $r \notin E'_\delta = E_\delta \cup [0, r_0]$ for some r_0. To make (3.8) useful we note that Rosenbloom (1962) showed that

$$E = \{r \colon B^2(r) > \log f_1(r)(\log\log f_1(r))^{2+\delta}\}$$

has finite logarithmic measure (cf. Erdös and Rényi [1, p. 50]). Without loss of generality we can assume that E is contained in the exceptional set E_δ of (3.1) and that

$$1 < B^2(r_k) < (\log \mu(r_k))^{1+\delta} \tag{3.9}$$

for all r_k, $k \geqslant k_0$. So, for such k, we have

$$(\log \mu(r_k))^{1/2} \leqslant C(r_k) = TB(r_k) \leqslant (\log \mu(r_k))^{1+\delta/2}$$

and by Lemma 2.2 (and the subsequent remark) by taking $N = [(\log \mu(r_k))^{1+\delta/2}] + 1$ we have

$$P\left(\max_{0 \leqslant \phi \leqslant 2\pi} \left| \sum_{|n - A(r_k)| \geqslant C(r_k)} a_n e^{in\phi} r_k^n \exp(i\theta_n t)\right| \right.$$
$$\left. \geqslant 2c_{q,B}(1+\delta)\, S(r_k)(\log\log \mu(r_k))^{1/2}\right)$$
$$\leqslant (\log \mu(r_k))^{-\beta}. \tag{3.10}$$

Now since $r_k \notin E_\delta$ we have

$$\mu(r_k) \leqslant S(r_k) \leqslant \mu(r_k)^{1/2} f_1(r_k)^{1/2}$$
$$\leqslant \mu(r_k)(\log \mu(r_k))^{1/8}(\log\log \mu(r_k))^{1/2+\delta/2}, \tag{3.11}$$

so, in particular,

$$\log \mu(r_k) \leqslant \log S(r_k) \leqslant 2 \log \mu(r_k) \tag{3.12}$$

for all k greater than some k_1. Choosing $\beta = 2$ and applying (3.10), (3.12) and the Borel–Cantelli lemma we have for almost every $0 < t < 2\pi$ that

$$\max_{0 \leqslant \phi \leqslant 2\pi} \left| \sum_{|n - A(r)| \geqslant C(r)} a_n e^{in\phi} r_k^n \exp(i\theta_n t)\right|$$
$$\leqslant (1+\delta)\, c_{\beta,q} S(r_k)(\log\log \mu(r_k))^{1/2} \tag{3.13}$$

for all $k > k(t)$. By (3.6), (3.8), (3.13) and the basic reduction (3.4) the proof of Theorem 2 is complete. ▮

IV. Best Possible

We will now sketch the proof that Theorem 1 cannot be improved as far as the exponent of $\log \mu(r)$ is concerned. This follows from the consideration of

$$f(z, \omega, t) = \sum_{n=0}^{\infty} \frac{1}{n!} e^{i\theta_n(\omega)} e^{i\theta_n t} z^n \tag{4.1}$$

where $\theta_n(\omega)$ are independent random variables which defined on a probability triple (Ω, F, v) and which are uniformly distributed on $[0, 2\pi]$.

The θ_n are the lacunary integers used earlier. The main observation is that for any $t \in [0, 2\pi]$ the random variables $X_n = e^{i\theta_n(\omega)} e^{i\theta_n t}$ are again i.i.d. bounded random variables with mean zero, so by (a mild generalization of) the assertion of Erdös and Rényi [1, p. 48], we have for all t

$$\lim_{r \to \infty} M(r, \omega, t)/\mu(r)(\log \mu(r))^{-\varepsilon + 1/4} = \infty \tag{4.2}$$

for all $\varepsilon > 0$ and all $\omega \in \Omega_{\varepsilon,t}$ with $\nu(\Omega_{\varepsilon,t}) = 1$. By Fubini's theorem the set $A = \{(\omega, t): \lim_{r \to \infty} M(r, \omega, t)/\mu(r)(\log \mu(r))^{-\varepsilon + 1/4} = \infty\}$ has $\nu \times (dt/2\pi)$ measure 1, and consequently we can find a section $A_{\omega_0} = A \cap \{(\omega, t): \omega = \omega_0\}$ which has $dt/2\pi$ measure 1. The choice $f(z, \omega_{0,t})$ thus provides an example of a function which shows the exponent of $\log \mu(r)$ in Theorem 1 cannot be taken as $\frac{1}{4} - \varepsilon$, $\varepsilon > 0$.

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