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A STEEP ASCENT OF THE FUNDAMENTAL THEOREM OF CALCULUS

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In *What is Mathematics?* Courant and Robbins give a proof of Euclid's theorem on the infinity of primes in which they use the fact that the zeta function diverges at $z = 1$. Speaking of their proof they say, "Of course, this is much more involved and sophisticated than the proof given by Euclid. But it has the fascination of a difficult ascent of a mountain peak which could be reached from the other side by a comfortable road." If one borrows this attitude and the fact (which is also indicated in *What is Mathematics?*) that the integral of x^n can be calculated without using differentiation, it is possible to give a proof of the fundamental theorem of calculus which analysis classes may find amusing.

By giving $C'[a, b]$ the norm

$$\|f\| = \sup|f(x)| + \sup|f'(x)|$$

we can easily see that the linear functional defined by

$$F(f) = \int_a^b f'(t)dt - f(b) + f(a)$$

is continuous. Now, as mentioned above, we may assume the usual formulas for the integral and derivative of x^n. This allows us to check that $F(p) = 0$ if p is a polynomial. However, in view of Weierstrass's Theorem (applied to f' for a given function f in $C'[a, b]$) and the Mean Value Theorem the polynomials are dense in $C'[a, b]$. Hence we have that F must vanish identically, which is just what the fundamental theorem says.