OPTIMAL GAMBLING SYSTEMS FOR FAVORABLE GAMES

L. BREIMAN
UNIVERSITY OF CALIFORNIA, LOS ANGELES

1. Introduction

Assume that we are hardened and unscrupulous types with an infinitely wealthy friend. We induce him to match any bet we wish to make on the event that a coin biased in our favor will turn up heads. That is, at every toss we have probability $p > 1/2$ of doubling the amount of our bet. If we are clever, as well as unscrupulous, we soon begin to worry about how much of our available fortune to bet at every toss. Betting everything we have on heads on every toss will lead to almost certain bankruptcy. On the other hand, if we bet a small, but fixed, fraction (we assume throughout that money is infinitely divisible) of our available fortune at every toss, then the law of large numbers informs us that our fortune converges almost surely to plus infinity. What to do?

More generally, let X be a random variable taking values in the set $I = \{1, \cdots, s\}$ such that $P\{X = i\} = p_i$ and let there be a class $\mathcal{C}$ of subsets A_j of I, where $\mathcal{C} = \{A_1, \cdots, A_r\}$, with $\bigcup_j A_j = I$, together with positive numbers $(o_1, \cdots, o_r)$. We play this game by betting amounts $\beta_1, \cdots, \beta_r$ on the events $\{X \in A_j\}$ and if the event $\{X = i\}$ is realized, we receive back the amount $\sum_{i \in A_j} \beta_j o_j$ where the sum is over all j such that $i \in A_j$. We may assume that our entire fortune is distributed at every play over the betting sets $\mathcal{C}$, because the possibility of holding part of our fortune in reserve is realized by taking A_1, say, such that $A_1 = I$, and $o_1 = 1$. Let S_n be the fortune after n plays; we say that the game is *favorable* if there is a gambling strategy such that almost surely $S_n \to \infty$. We give in the next section a simple necessary and sufficient condition for a game to be favorable.

How much to bet on the various alternatives in a sequence of independent repetitions of a favorable game depends, of course, on what our goal utility is. There are two criterions, among the many possibilities, that seem pre-eminently reasonable. One is the minimal time requirement, that is, we fix an amount x we wish to win and inquire after that gambling strategy which will minimize the expected number of trials needed to win or exceed x. The other is a magnitude condition; we fix at n the number of trials we are going to play and examine the size of our fortune after the n plays.

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In this work, we are especially interested in the asymptotic point of view. We show that in the long run, from either of the two above criterions, there is one strategy Λ^* which is optimal. This strategy is found as that system of betting (essentially unique) which maximizes $E(\log S_n)$. The reason for this result is heuristically clear. Under reasonable betting systems S_n increases exponentially and maximizing $E(\log S_n)$ maximizes the rate of growth.

In the second section we investigate the nature of Λ^*. It is a conservative policy which consists in betting *fixed* fractions of the available fortune on the various A_j. For example, in the coin-tossing game Λ^* is: bet a fraction $p - q$ of our fortune on heads at every game. It is also, in general, a policy of diversification involving the placing of bets on many of the A_j rather than the single one with the largest expected return.

The minimal expected time property is covered in the third section. We show, by an examination of the excess in Wald's formula, that the desired fortune x becomes infinite, that the expected time under Λ^* to amass x becomes less than that under any other strategy.

Section four is involved with the magnitude problem. The content here is that Λ^* magnitudewise, does as well as any other strategy, and that if one picks a policy which in the long run does not become close to Λ^*, then we are asymptotically infinitely worse off.

Finally, in section five, we discuss the finite (nonasymptotic) case for the coin-tossing game. We have been unsuccessful in our efforts to find a strategy which minimizes the expected time for x fixed, but we state a conjecture which expresses a moderate faith in the simplicity of things. It is not difficult, however, to find a strategy which maximizes $P\{S_n \geqq x\}$ for fixed n, x and we state the results with only a scant indication of proof, and then launch into a comparison with the strategy Λ^* for large n.

The conclusion of these investigations is that the strategy Λ^* seems by all reasonable standards to be asymptotically best, and that, in the finite case, it is suboptimal in the sense of providing a uniformly good approximation to the optimal results.

Since completing this work we have been allowed to examine the most significant manuscript of L. Dubins and L. J. Savage [1], which will soon be published. Although gambling has been associated with probability since its birth, only quite recently has the question of gambling systems optimal with respect to some goal utility been investigated carefully. To the beautiful and deep results of Dubins and Savage, upon which work was commenced in 1956, must be given priority as the first to formulate systematically and solve the problems of optimal gambling strategies. We strongly recommend their work to every student of probability theory.

Although our original impetus came from a different source, and although their manuscript is almost wholly concerned with unfavorable and fair games, there are a few small areas of overlap which I should like to point out and acknowledge priority. Dubins and Savage did, of course, formulate the concept of a favorable game. For these games they considered the class of "fractionalizing strategies," which consist in betting a fixed fraction of one's fortune at every play, and noticed the interesting phenomenon that there was a critical fraction such that if one bets a fixed fraction less than this critical value, then $S_n \to \infty$ a.s. and if one bets a fixed fraction greater than this critical value, then $S_n \to 0$ a.s. In addition, our proposition 3 is an almost exact duplication of one of their theorems. In their work, also, will be found the solution to maximizing $P\{S_n \geqq x\}$ for an unfavorable game, and it is interesting to observe here the abrupt discontinuity in strategies as the game changes from unfavorable to favorable.

My original curiosity concerning favorable games dates from a paper of J. L. Kelly, Jr. [2] in which there is an intriguing interpretation of information theory from a gambling point of view. Finally, some of the last section, in problem and solution, is closely related to the theory of dynamic programming as originated by R. Bellman [3].

2. The nature of Λ^*

We introduce some notation. Let the outcome of the kth game be X_k and $R_n = (X_n, \cdots, X_1)$. Take the initial fortune S_0 to be unity, and S_n the fortune after n games. To specify a strategy Λ we specify for every n, the fractions $[\lambda_1^{(n+1)}, \cdots, \lambda_r^{(n+1)}] = \bar{\lambda}_{n+1}$ of our available fortune after the nth game, S_n, that we will bet on alternative $A_1, \cdots, A_r$ in the $(n+1)$st game. Hence

$$\sum_{j=1}^{r} \lambda_j^{(n+1)} = 1. \tag{2.1}$$

Note that $\bar{\lambda}^{(n+1)}$ may depend on R_n. Denote $\Lambda = (\bar{\lambda}_1, \bar{\lambda}_2, \cdots)$. Define the random variables V_n by

$$V_n = \sum_{i \in A_j} \lambda_j^{(n)} o_j, \qquad X_n = i, \tag{2.2}$$

so that $S_{n+1} = V_{n+1} S_n$. Let $W_n = \log V_n$, so we have

$$\log S_n = W_n + \cdots + W_1. \tag{2.3}$$

To define Λ^*, consider the set of vectors $\bar{\lambda} = (\lambda_1, \cdots, \lambda_r)$ with r nonnegative components such that $\lambda_1 + \cdots + \lambda_r = 1$ and define a function $W(\bar{\lambda})$ on this space $\mathfrak{F}$ by

$$W(\bar{\lambda}) = \sum_i p_i \log \Big(\sum_{i \in A_j} \lambda_j o_j \Big). \tag{2.4}$$

The function $W(\bar{\lambda})$ achieves its maximum on $\mathfrak{F}$ and we denote $W = \max_{\bar{\lambda} \in \mathfrak{F}} W(\bar{\lambda})$.

PROPOSITION 1. *Let $\bar{\lambda}^{(1)}$, $\bar{\lambda}^{(2)}$ be in $\mathfrak{F}$ such that $W = W(\bar{\lambda}^{(1)}) = W(\bar{\lambda}^{(2)})$, then for all i, we have $\sum_{i \in A_j} \lambda_j^{(1)} o_j = \sum_{i \in A_j} \lambda_j^{(2)} o_j$.*

PROOF. Let α, β be positive numbers such that $\alpha + \beta = 1$. Then if $\bar{\lambda} = \alpha \bar{\lambda}^{(1)} + \beta \bar{\lambda}^{(2)}$, we have $W(\bar{\lambda}) \leqq W$. But by the concavity of log

$$W(\bar{\lambda}) \geqq \alpha W(\bar{\lambda}^{(1)}) + \beta W(\bar{\lambda}^{(2)}) \tag{2.5}$$

with equality if and only if the conclusion of the proposition holds.

Now let $\bar{\lambda}^*$ be such that $W = W(\bar{\lambda}^*)$ and define Λ^* as $(\bar{\lambda}^*, \bar{\lambda}^*, \cdots)$. Although $\bar{\lambda}^*$ may not be unique, the random variables $W_1^*, W_2^*, \cdots$ arising from Λ^* are by proposition 1 uniquely defined, and form a sequence of independent, identically distributed random variables.

Questions of uniqueness and description of $\bar{\lambda}^*$ are complicated in the general case. But some insight into the type of strategy we get using Λ^* is afforded by

PROPOSITION 2. *Let the sets $A_1, \cdots, A_r$ be disjoint, then no matter what the odds o_j are, $\bar{\lambda}^*$ is given by $\lambda_j^* = P\{X \in A_j\}$.*

The proof is a simple computation and is omitted.

From now on we restrict attention to favorable games and give the following criterion.

PROPOSITION 3. *A game is favorable if and only if $W > 0$.*

PROOF. We have

$$\log S_n^* = \sum_1^n W_k^*. \tag{2.6}$$

If $W = EW_k^*$ is positive, then the strong law of large numbers yields $S_n^* \to \infty$ a.s. Conversely, if there is a strategy Λ such that $S_n \to \infty$ a.s. we use the result of section 4, which says that for any strategy Λ, $\lim_n S_n/S_n^*$ exists a.s. finite. Hence $S_n^* \to \infty$ a.s. and therefore $W \geqq 0$. Suppose $W = 0$, then the law of the iterated logarithm comes to our rescue and provides a contradiction to $S_n^* \to \infty$.

3. The asymptotic time minimization problem

For any strategy Λ and any number $x > 1$, define the random variable $T(x)$ by

$$T(x) = \{\text{smallest } n \text{ such that } S_n \geqq x\}, \tag{3.1}$$

and $T^*(x)$ the corresponding random variable using the strategy Λ^*. That is, $T(x)$ is the number of plays needed under Λ to amass or exceed the fortune x. This section is concerned with the proof of the following theorem.

THEOREM 1. *If the random variables $W_1^*, W_2^*, \cdots$ are nonlattice, then for any strategy*

$$\lim_{x \to \infty} [ET(x) - ET^*(x)] = \frac{1}{W} \sum_1^\infty (W - EW_n) \tag{3.2}$$

and there is a constant α, independent of Λ and x such that

$$ET^*(x) - ET(x) \leqq \alpha. \tag{3.3}$$

Notice that the right side of (3.2) is always nonnegative and is zero only if Λ is equivalent to Λ^* in the sense that for every n, we have $W_n = W_n^*$. The reason for the restriction that W_n^* be nonlattice is fairly apparent. But as this restriction is on $\log V_n^*$ rather than on V_n^* itself, the common games with rational values of the odds o_j and probabilities p_i usually will be nonlattice. For instance, a little number-theoretic juggling proves that in the coin-tossing case the countable set of values of p for which W_n^* is lattice consists only of irrationals.

The proof of the above theorem is long and will be carried out in a sequence of propositions. The heart is an asymptotic estimate of the excess in Wald's identity [4].

PROPOSITION 4. *Let $X_1, X_2, \cdots$ be a sequence of identically distributed, independent nonlattice random variables with $0 < EX_1 < \infty$. Let $Y_n = X_1 + \cdots + X_n$. For any real numbers x, ξ, with $\xi > 0$, let $F_x(\xi) = P\{\text{first } Y_n \geqq x \text{ is } < x + \xi\}$. Then there is a continuous distribution $G(\xi)$ such that for every value of ξ,*

$$\lim_{x \to \infty} F_x(\xi) = G(\xi). \tag{3.4}$$

PROOF. The above statement is contained in known results concerning the renewal theorem. If $X_1 > 0$ a.s. and has the distribution function F, it is known (see, for example, [5]) that $\lim_{x\to\infty} F_x(\xi) = (1/EX_1) \int_0^\xi [1 - F(t)]\, dt$. If X_1 is not positive, we use a device due to Blackwell [6]. Define the integer-valued random variables $n_1 < n_2 < \cdots$ by $n_1 = \{\text{first } n \text{ such that } X_1 + \cdots + X_n > 0\}$, $n_2 = \{\text{first } n \text{ such that } X_{n_1+1} + \cdots + X_n > 0\}$, and so forth. Then the random variables $X_1' = X_1 + \cdots + X_{n_1}$, $X_2' = X_{n_1+1} + \cdots + X_{n_2}$, $\cdots$ are independent, identically distributed, positive, and $EX_1' < \infty$ (see [6]). Letting $Y_n' = X_1' + \cdots + X_n'$, note that $P\{\text{first } Y_n \geqq x \text{ is } < x + \xi\} = P\{\text{first } Y_n' \geqq x \text{ is } < x + \xi\}$, which completes the proof.

We find it useful to transform this problem by defining for any strategy Λ, a random variable $N(y)$,

$$N(y) = \{\text{smallest } n \text{ such that } W_n + \cdots + W_1 \geqq y\} \tag{3.5}$$

with $N^*(y)$ the analogous thing for Λ^*. To prove (3.2) we need to prove

$$\lim_{y \to \infty} [EN(y) - EN^*(y)] = \frac{1}{W} \sum_1^\infty (W - EW_n), \tag{3.6}$$

and we use a result very close to Wald's identity.

PROPOSITION 5. *For any strategy Λ such that $S_n \to \infty$ a.s. and any y*

$$EN(y) = \frac{1}{W} E\left\{\sum_{k=1}^{N(y)} [W - E(W_k|R_{k-1})]\right\} + \frac{1}{W} E\left[\sum_{k=1}^{N(y)} W_k\right]. \tag{3.7}$$

PROOF. The above identity is derived in a very similar fashion to Doob's derivation [6] of Wald's identity. The difficult point is an integrability condition and we get around this by using, instead of the strategy Λ, a modification Λ_J which consists in using Λ for the first J plays and then switching to $\bar{\Lambda}^*$. The condition $S_n \to \infty$ a.s. implies that none of the W_k may take on the value $-\infty$ and that $N(y)$ is well defined. Let $N_J(y)$ be the random variable analogous to $N(y)$ under Λ_J and $W_k^{(J)}$ to W_k. Define a sequence of random variables Z_n by

$$Z_n = \sum_1^n [W_k^{(J)} - E(W_k^{(J)}|R_{k-1})]. \tag{3.8}$$

This sequence is a martingale with $EZ_n = 0$. By Wald's identity, $EN_J(y) < \infty$ and it is seen that the conditions of the optional sampling theorem ([7], theorem 2.2–C_3) are validated with the conclusion that $EZ_{N_J} = 0$. Therefore

$$WEN_J = E\left[\sum_{k=1}^{N_J} W\right] \tag{3.9}$$
$$= E\left\{\sum_{k=1}^{N_J} [W - E(W_k^{(J)}|R_{k-1})]\right\} + E\left[\sum_{k=1}^{N_J} W_k^{(J)}\right]$$
$$= E\left\{\sum_{k=1}^{\min(N,J)} [W - E(W_k|R_{k-1})]\right\} + E\left[\sum_{k=1}^{N_J} W_k^{(J)}\right].$$

The second term on the right satisfies

$$y \leqq E\left[\sum_{k=1}^{N_J} W_k^{(J)}\right] \leqq y + \alpha, \tag{3.10}$$

where $\alpha = \max_j (\log o_j)$. Hence, if $EN = \infty$, then $\lim_J EN_J = \infty$, so that $E\{\sum_1^{N_J} [W - E(W_k|R_{k-1})]\} = \infty$ and (3.7) is degenerately true. Now assume that $EN < \infty$, and let $J \to \infty$. The first term on the right in (3.9) converges to $E\{\sum_1^N [W - E(W_k|R_{k-1})]\}$ monotonically. The random variables $\sum_1^{N_J} W_k^{(J)}$ converge a.s. to $\sum_1^N W_k$ and are bounded below and above by y and $y + \alpha$ so that the expectations converge. It remains to show that $\lim_J EN_J = EN$. Since

$$EN_J = \int_{\{N \leqq J\}} N\, dP + \int_{\{N > J\}} N_J\, dP, \tag{3.11}$$

we need to show that the extreme right term converges to zero. Let

$$U_J = \sum_1^J W_k,\ N(U_J) = \{\text{first } n \text{ such that} \sum_{J+1}^{n+J} W_k^{(J)} \geqq y - U_J\} \tag{3.12}$$

so that

$$\int_{\{N > J\}} N_J\, dP = JP(N > J) + \int_{\{N > J\}} N(U_J)\, dP. \tag{3.13}$$

Since $EN < \infty$, we have $\lim_J JP\{N > J\} = 0$. We write the second term as $E\{E[N(U_J)|U_J]|N > J\}\, P\{N > J\}$. By Wald's identity,

$$E[N(U_J)|U_J] \leqq \frac{y - U_J + \alpha}{W}. \tag{3.14}$$

On the other hand, since the most we can win at any play is α, the inequality

$$N \geqq \frac{y - U_J}{\alpha} + J \tag{3.15}$$

holds on the set $\{N > J\}$. Putting together the pieces,

$$\int_{\{N > J\}} N(U_J)\, dP \leqq \frac{\alpha}{W} \int_{\{N > J\}} (N - J)\, dP + \frac{\alpha}{W} P(N > J). \tag{3.16}$$

The right side converges to zero and the proposition is proven.

If we subtract from (3.7) the analogous result for Λ^* we get

$$EN(y) - EN^*(y) \tag{3.17}$$
$$= \frac{1}{W} E\left\{\sum_{k=1}^{N} [W - E(W_k|R_{k-1})]\right\} + \frac{1}{W} E\left[\sum_{k=1}^{N} W_k - \sum_{k=1}^{N^*} W_k^*\right].$$

This last result establishes inequality (3.3) of the theorem. As we let $y \to \infty$, then $N(y) \to \infty$ a.s. and we see that

$$\lim_{y \to \infty} E\left\{\sum_{k=1}^{N} [W - E(W_k | R_{k-1})]\right\} = \sum_{k=1}^{\infty} (W - EW_k). \tag{3.18}$$

By proposition 4, the distribution F_y^* of $\sum_1^{N^*} W_k^* - y$ converges, as $y \to \infty$, to some continuous distribution F^* and we finish by proving that the distribution F_y of $\sum_1^N W_k - y$ also converges to F^*.

PROPOSITION 6. *Let Y_n, ϵ_n be two sequences of random variables such that $Y_n \to \infty$, $Y_n + \epsilon_n \to \infty$ a.s. If Z is any random variable, if $\epsilon = \sup_{n \geq 1} |\epsilon_n|$, and if we define*

$$H_y(\xi) = P\{\text{first } Y_n \geqq Z + y \text{ is } < Z + y + \xi\}, \tag{3.19}$$

$$D_y(\xi) = P\{\text{first } Y_n + \epsilon_n \geqq Z + y \text{ is } < Z + y + \xi\}, \tag{3.20}$$

then for any $u > 0$,

(3.21)

$$F_{z+u}(\xi - 2u) - P\{\epsilon \geqq u\} \leqq D_y(\xi) \leqq D_y(\xi) \leqq H_{y-u}(\xi + 2u) + P\{\epsilon \geqq u\}.$$

PROOF.

(3.22)

$$\begin{aligned} D_y(\xi) &\leqq P\{\text{first } Y_n + \epsilon_n \geqq Z + y \text{ is } < Z + y + \xi, \epsilon < u\} + P\{\epsilon \geqq u\} \\ &\leqq P\{\text{first } Y_n > Z + y - u \text{ is } < Z + y + \xi + u, \epsilon < u\} + P\{\epsilon \geqq u\} \\ &\leqq H_{y-u}(\xi + 2u) + P\{\epsilon \geqq u\}. \end{aligned}$$

(3.23)

$$\begin{aligned} D_y(\xi) &\geqq P\{\text{first } Y_n + \epsilon_n \geqq Z + y \text{ is } < Z + y + \xi, \epsilon < u\} \\ &\geqq P\{\text{first } Y_n \geqq Z + y + u \text{ is } < Z + y + \xi - u, \epsilon < u\} \\ &\geqq H_{y+u}(\xi - 2u) - P\{\epsilon \geqq u\}. \end{aligned}$$

PROPOSITION 7. *Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed nonlattice random variables, $0 < EX_1 < \infty$, with $Y_n = X_1 + \cdots + X_n$. If Z is any random variable independent of $X_1, X_2, \cdots$, G the limiting distribution of proposition 4, and*

$$F_{y,Z}(\xi) = P\{\text{first } Y_n \geqq Z + y \text{ is } < Z + y + \xi\}, \tag{3.24}$$

then $\lim_y F_{y,Z}(\xi) = G(\xi)$.

PROOF.

$$\begin{aligned} F_{y,Z}(\xi) &= E[P\{\text{first } Y_n \geqq Z + y \text{ is } < Z + y + \xi | Z\}] \\ &= E[F_{y+Z}(\xi)], \end{aligned} \tag{3.25}$$

where $F_y(\xi) = P\{\text{first } Y_n \geqq y \text{ is } < y + \xi\}$. But $\lim_y F_{y+Z}(\xi) = G(\xi)$ a.s. which, together with the boundedness of $F_{y+Z}(\xi)$, establishes the result.

We start putting things together with

PROPOSITION 8. *Let $\sum_1^n W_k - \sum_1^n W_k^*$ converge a.s. to an everywhere finite limit. If the W_k^* are nonlattice, if $F_y(\xi)$ is the distribution function for $\sum_1^N W_k - y$, then* $\lim_y F_y(\xi) = F^*(\xi)$.

PROOF. Fix m, let

$$Z_m = -\sum_1^{m-1} W_k, \qquad \epsilon_{m,n} = \sum_m^n W_k - \sum_m^n W_k^*, \qquad \epsilon_m = \sup_n |\epsilon_{m,n}|, \tag{3.26}$$

and by assumption $\epsilon_m \to 0$ a.s. Now

$$\begin{aligned} F_y(\xi) &= P\{\text{first } \sum_1^n W_k \geqq y \text{ is } < y + \xi\} \\ &= P\{\text{first } \left(\sum_m^n W_k^* + \epsilon_{m,n}\right) \geqq Z_m + y \text{ is } < Z_m + y + \xi\}. \end{aligned} \tag{3.27}$$

If

$$H_y(\xi) = P\{\text{first } \sum_m^n W_k^* \geqq Z_m + y \text{ is } < Z_m + y + \xi\}, \tag{3.28}$$

then by proposition 6, for any $u > 0$,

$$H_{y+u}(\xi - 2u) - P\{\epsilon_m \geqq u\} \leqq F_y(\xi) \leqq H_{y-u}(\xi + 2u) + P\{\epsilon_m \geqq u\}. \tag{3.29}$$

Letting $y \to \infty$ and applying proposition 7,

(3.30)

$$F^*(\xi - 2u) - P\{\epsilon_m \geqq u\} \leqq \varliminf_y F_y(\xi) \leqq \varlimsup_y F_y(\xi) \leqq F^*(\xi + 2u) + P\{\epsilon_m \geqq u\}.$$

Taking first $m \to \infty$ and then $u \to 0$ we get

$$\varliminf_y F_y(\xi) = \varlimsup_y F_y(\xi) = F^*(\xi). \tag{3.31}$$

To finish the proof, we invoke theorems 2 and 3 of section 4. The content we use is that if $\sum_1^n W_k - \sum_1^n W_k^*$ does not converge a.s. to an everywhere finite limit, then $\sum_1^\infty [W - E(W_k|R_{k-1})] = +\infty$ on a set of positive probability. Therefore, if the conditions of propositions 5 and 8 are not validated, then by (3.17) both sides of (3.2) are infinite. Thus the theorem is proved.

3. Asymptotic magnitude problem

The main results of this section can be stated roughly as: asymptotically, S_n^* is as large as the S_n provided by any strategy Λ, and if Λ is not asymptotically close to Λ^*, then S_n^* is infinitely larger than S_n. The results are valid whether or not the games are favorable.

THEOREM 2. *Let Λ be any strategy leading to the fortune S_n after n plays. Then $\lim_n S_n/S_n^*$ exists a.s. and $E(\lim_n S_n/S_n^*) \leqq 1$.*

For the statement of theorem 3 we need

DEFINITION. *Λ is a nonterminating strategy if there are no values of $\bar{\lambda}_n$ such that $\sum_{i \in A_j} \lambda_i^{(n)} o_j = 0$, for any n.*

THEOREM 3. *If Λ is a nonterminating strategy, then almost surely*

$$\sum_1^\infty [W - E(W_k|R_{k-1})] = \infty \Leftrightarrow \lim_n \frac{S_n^*}{S_n} = \infty. \tag{4.1}$$

PROOFS. We present the theorems together as their proofs are similar and hinge on the martingale theorems. For every n

$$E\left(\frac{S_n}{S_n^*}\middle|R_{n-1}\right) = E\left(\frac{V_n}{V_n^*}\middle|R_{n-1}\right)\frac{S_{n-1}}{S_{n-1}^*}. \tag{4.2}$$

If we prove that $E(V_n/V_n^*|R_{n-1}) \leqq 1$ a.s., then S_n/S_n^* is a decreasing semi-martingale with $\lim_n S_n/S_n^*$ existing a.s. and

$$E \lim_n \frac{S_n}{S_n^*} \leqq E\frac{S_0}{S_0^*} = 1. \tag{4.3}$$

By the definition of Λ^*, for every $\epsilon > 0$,

$$E\{\log[(1-\epsilon)V_n^* + \epsilon V_n] - \log V_n^*|R_{n-1}\} \leqq 0. \tag{4.4}$$

Manipulating gives

$$\frac{1}{\epsilon}E\left[\log\left(1 + \frac{\epsilon}{1-\epsilon}\frac{V_n}{V_n^*}\right)\middle|R_{n-1}\right] \leqq \frac{1}{\epsilon}\log\frac{1}{1-\epsilon}. \tag{4.5}$$

By Fatou's lemma, as $\epsilon \to 0$

$$\begin{aligned} E\left(\frac{V_n}{V_n^*}\middle|R_{n-1}\right) &= E\left[\underline{\lim}\left(\frac{1}{\epsilon}\log 1 + \frac{\epsilon}{1-\epsilon}\frac{V_n}{V_n^*}\right)\middle|R_{n-1}\right] \\ &\leqq \underline{\lim}\,\frac{1}{\epsilon}\log\frac{1}{1-\epsilon} = 1. \end{aligned} \tag{4.6}$$

Theorem 3 resembles a martingale theorem given by Doob ([6], pp. 323–324), but integrability conditions get in our way and force some deviousness. Fix a number $M > 0$ and take A to be the event $\{W - E(W_n|R_{n-1}) \geqq M \text{ i.o.}\}$. If $p = \min_i p_i$, then $E(W_n^* - W_n|R_{n-1}) \geqq M$ implies $P\{W_n^* - W_n \geqq M|R_{n-1}\} \geqq p$. By the conditional version of the Borel-Cantelli lemma ([7], p. 324), the set on which $\sum_1^\infty P\{W_n^* - W_n \geqq M|R_{n-1}\} = \infty$ and the set $\{W_n^* - W_n \geqq M \text{ i.o.}\}$ are a.s. the same. Therefore, a.s. on A, we have $W_n^* - W_n \geqq M$ i.o. and $\log(S_n^*/S_n) = \sum_1^n (W_k^* - W_k)$ cannot converge. We conclude that both sides of (4.1) diverge a.s. on A.

Starting with a strategy Λ, define an amended strategy Λ_M by: if $W - E(W_n|R_{n-1}) < M$, use Λ on the nth play, otherwise use Λ^* on the nth play. The random variables

$$U_n = \log\frac{S_n^*}{S_n^{(M)}} - \sum_1^n [W - E(W_k^{(M)}|R_{k-1})] \tag{4.7}$$

form a martingale sequence with

$$U_n - U_{n-1} = W_n^* - W_n^{(M)} - [W - E(W_n^{(M)}|R_{n-1})]. \tag{4.8}$$

For Λ_M, we have $E(W_n^* - W_n^{(M)}|R_{n-1}) < M$, leading to the inequalities,

$$\sup (W_n^* - W_n^{(M)}) \leqq \frac{M}{p}, \qquad U_n - U_{n-1} \leqq \frac{M}{p}. \tag{4.9}$$

On the other side, if

$$\alpha = \min_j \log \Big(\sum_{i \in A_j} \lambda_j^* o_j\Big), \qquad \beta = \max_j \log o_j, \tag{4.10}$$

then $U_n - U_{n-1} \geqq \alpha - \beta - M$. These bounds allow the use of a known martingale theorem ([7], pp. 319–320) to conclude that $\lim_n U_n$ exists a.s. whenever one of $\overline{\lim}\, U_n < \infty$, $\underline{\lim}\, U_n > -\infty$ is satisfied. This implies the statement

$$\lim \frac{S_n^*}{S_n^{(M)}} < \infty \Leftrightarrow \sum_1^\infty [W - E(W_k^{(M)}|R_{k-1})] < \infty. \tag{4.11}$$

However, on the complement of the set A the convergence or divergence of the above expressions involves the convergence or divergence of the corresponding quantities in (4.1) which proves the theorem.

Corollary 1. *If for some strategy Λ, we have $\sum_1^\infty [W - E(W_k|R_{k-1})] = \infty$ with probability $\gamma > 0$, then for every $\epsilon > 0$, there is a strategy $\hat{\Lambda}$ such that with probability at least $\gamma - \epsilon$, $\overline{\lim}\, S_n/\hat{S}_n = 0$ and except for a set of probability at most ϵ, $\overline{\lim}\, S_n/\hat{S}_n \leqq 1$.*

Proof. Let E be the set on which $\lim S_n/S_n^* = 0$, with $P\{E\} = \gamma$. For any $\epsilon > 0$, for N sufficiently large, there is a set E_N, measurable with respect to the field generated by R_N such that $P\{E_N \Delta E\} < \epsilon$, where Δ denotes the symmetric set difference. Define $\hat{\Lambda}$ as follows: if $n < N$, use Λ, if R_n, with $n \geqq N$, is such that the first N outcomes $(X_1, \cdots, X_N)$ is not in E_N, use Λ, otherwise use Λ^*. On E_N, we have $\sum_1^\infty [W - E(\hat{W}_k|R_{k-1})] < \infty$, hence $\lim \hat{S}_n/S_n^* > 0$ so that $\lim S_n/\hat{S}_n = 0$ on $E_N \cap E$. Further, $P\{E_N \cap E\} \geqq P\{E\} - \epsilon = \gamma - \epsilon$. On the complement of E_N, we have $S_n = \hat{S}_n$, leading to $\overline{\lim}\, S_n/\hat{S}_n \leqq 1$, except for a set with probability at most ϵ.

5. Problems with finite goals in coin tossing

In this section we consider first the problem: fix an integer $n > 0$, and two numbers $y > x > 0$, find a strategy which maximizes $P\{S_n \geqq y|S_0 = x\}$. In this situation, then, only n plays of the game are allowed and we wish to maximize the probability of exceeding a certain return. We will also be interested in what happens as n, y become large. By changing the unit of money, note that

$$\sup P\{S_n \geqq y|S_0 = x\} = \sup P\left\{S_n \geqq 1|S_0 = \frac{x}{y}\right\}, \tag{5.1}$$

where the supremum is over all strategies. Thus, the problem reduces to the unit interval, and we may evidently translate back to the general case if we find an optimum strategy in the reduced case. Define, for $\xi \geqq 0$, $n \geqq 1$,

(5.2)
$$\phi_n(\xi) = \begin{cases} \sup P\{S_n \geqq 1 | S_0 = \xi\}, & \xi < 1, \\ 1, & \xi \geqq 1, \end{cases}$$

and

(5.3)
$$\phi_0(\xi) = \begin{cases} 0, & \xi < 1, \\ 1, & \xi \geqq 1. \end{cases}$$

In addition $\phi_n(\xi)$ satisfies

(5.4)
$$\begin{aligned} \phi_n(\xi) &= \sup E[P\{S_n \geqq 1 | S_1, S_0\} | S_0 = \xi] \\ &\leqq \sup E[\phi_{n-1}(S_1) | S_0 = \xi] \\ &\leqq \sup_{0 \leqq z \leqq \xi} [p\phi_{n-1}(\xi + z) + q\phi_{n-1}(\xi - z)]. \end{aligned}$$

To find $\phi_n(\xi)$ and an optimal strategy, we define functions $\hat{\phi}_n(\xi)$ by

(5.5)
$$\hat{\phi}_0(\xi) = \phi_0(\xi), \qquad \hat{\phi}_n(\xi) = \sup_{0 \leqq z \leqq \xi} [p\hat{\phi}_{n-1}(\xi + z) + q\hat{\phi}_{n-1}(\xi - z)]$$

having the property $\phi_n(\xi) \leqq \hat{\phi}_n(\xi)$, for all n, ξ. If we can find a strategy Λ such that under Λ we have $\hat{\phi}_n(\xi) = P\{S_n \geqq 1 | S_0 = \xi\}$, then, evidently, Λ is optimum, and $\hat{\phi}_n = \phi_n$. But, if for every $n \geqq 1$, and ξ there is a $z_n(\xi)$, with $0 \leqq z_n(\xi) \leqq \xi$, such that

(5.6)
$$\hat{\phi}_n(\xi) = p\hat{\phi}_{n-1}[\xi + z_n(\xi)] + q\hat{\phi}_{n-1}[\xi - z_n(\xi)],$$

then we assert that the optimum strategy is Λ defined as: if there are m plays left and we have fortune ξ, bet the amount $z_m(\xi)$. Because, suppose that under Λ, for $n = 0, 1, \cdots, m$ we have $\hat{\phi}_n(\xi) = P\{S_n \geqq 1 | S_0 = \xi\}$, then

(5.7)
$$\begin{aligned} P\{S_{m+1} \geqq 1 | S_0 = \xi\} &= E[P\{S_{m+1} \geqq 1 | S_1, S_0\} | S_0 = \xi] \\ &= E[\hat{\phi}_m(S_1) | S_0 = \xi] = \hat{\phi}_{m+1}(\xi). \end{aligned}$$

Hence, we need only solve recursively the functional equation (5.5) and then look for solutions of (5.6) in order to find an optimal strategy. We will not go through the complicated but straightforward computation of $\hat{\phi}_n(\xi)$. It can be described by dividing the unit interval into 2^n equal intervals $I_1, \cdots, I_{2n}$ such that $I_k = [k/2^n, (k+1)/2^n]$. In tossing a coin with $P\{H\} = p$, rank the probabilities of the 2^n outcomes of n tosses in descending order $P_1 \geqq P_2 \geqq \cdots \geqq P_{2^n}$, that is, $P_1 = p^n$, $p_{2^n} = q^n$. Then, as shown in figure 1,

(5.8)
$$\phi_n(\xi) = \sum_{j<k} P_j, \qquad \xi \in I_k,$$

Note that if $p > 1/2$, then $\lim_n \phi_n(\xi) = 1$, with $\xi > 0$; and in the limiting case $p = 1/2$, then $\lim_n \phi_n(\xi) = \xi$, with $\xi \leqq 1$, in agreement with the Dubins-Savage result [2].

There are many different optimum strategies, and we describe the one which seems simplest. Divide the unit interval into $n + 1$ subintervals $I_0^{(n)}, \cdots, I_n^{(n)}$, such that the length of $I_k^{(n)}$ is $2^{-n}\binom{n}{k}$ where the $\binom{n}{k}$ are binomial coefficients. On

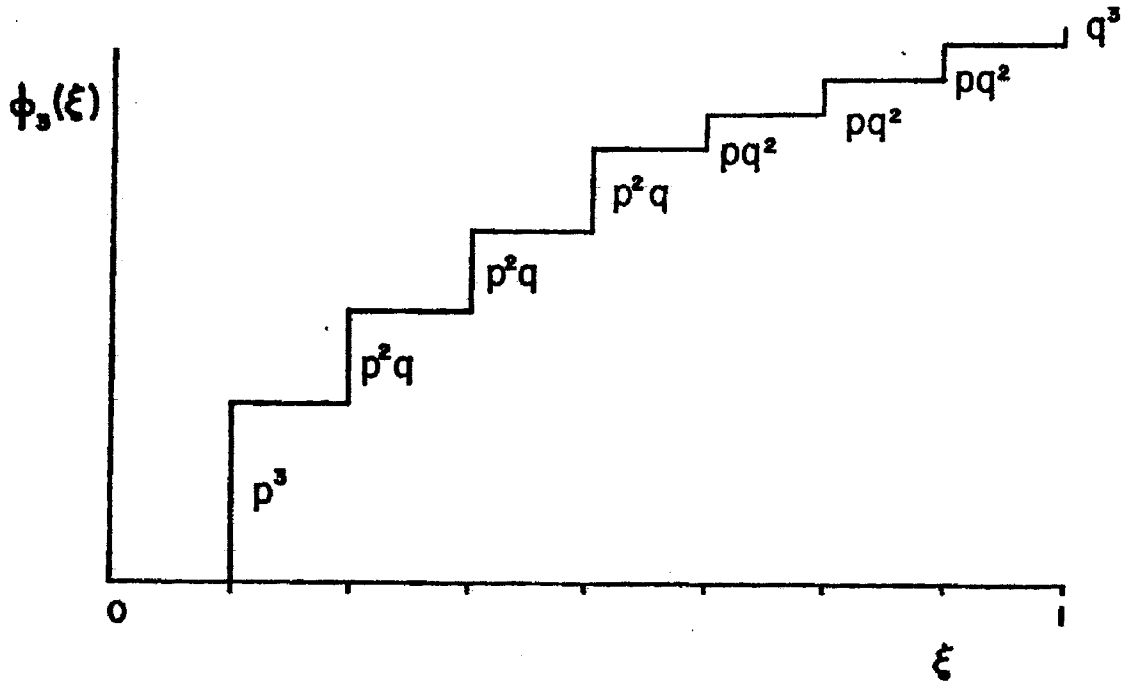

FIGURE 1
Graph of $\phi_n(\xi)$ for the case $n = 3$.

each $I_k^{(n)}$ as base, erect a 45°–45° isosceles triangle. Then the graph of $z_{n+1}(\xi)$ is formed by the sides of these triangles, as shown in figure 2. Roughly, this strategy calls for a preliminary "jockeying for position," with the preferred positions with m plays remaining being the midpoints of the intervals $I_k^{(m)}$. Notice that the endpoints of the intervals $\{I_k^{(n)}\}$ form the midpoints of the intervals $\{I_k^{(n-1)}\}$. So that if with n plays remaining we are at a midpoint of $\{I_k^{(n)}\}$, then at all remaining plays we will be at midpoints of the appropriate system of intervals. Very interestingly, this strategy is independent of the values of p so

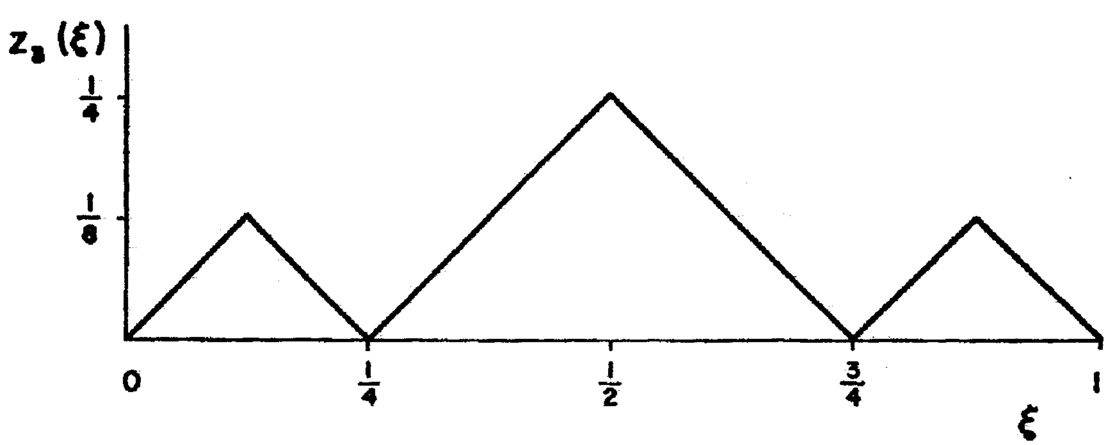

FIGURE 2
Graph of $z_{n+1}(\xi)$ for the case $n = 3$.

long as $p > 1/2$. The strategy Λ^* in this case is: bet a fraction $p - q$ of our fortune at every play. Let $\phi_n^*(\xi) = P\{S_n^* \geqq 1 | S_0 = \xi\}$. In light of the above remark, the following result is not without gratification.

THEOREM 4. $\lim_n \sup_\xi [\phi_n(\xi) - \phi_n^*(\xi)] = 0$.

PROOF. The proof is somewhat tedious, using the central limit theorem and tail estimates. However, some interesting properties of $\phi_n(\xi)$ will be discovered along the way. Let $P\{k|1/2\}$ be the probability of k or fewer tails in tossing a fair coin n times, $P\{k|p\}$ the probability of k or fewer tails in n tosses of a coin with $P\{H\} = p$. If $\xi = P\{k|1/2\} + 2^{-n}$, note that $\phi_n(\xi -) = P\{k|p\}$. Let $\sigma = \sqrt{pq}$, by the central limit theorem, if $\xi_{t,n} = P\{qn + t\sigma\sqrt{n}|1/2\} + 2^{-n}$, then

$$\lim_n \phi_n(\xi_{t,n} -) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2}\, dx, \tag{5.9}$$

uniformly in t. Thus, if we establish that

$$\lim_n \phi_n^*(\xi_{t,n}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2}\, dx \tag{5.10}$$

uniformly for t in any bounded interval, then by the monotonicity of $\phi_n(\xi)$, $\phi_n^*(\xi)$, the theorem will follow.

By definition,

$$\begin{aligned} \phi_n^*(\xi) &= P\{W_1^* + \cdots + W_n^* \geqq 0 | W_0 \\ &= \log \xi\} = P\{W_1^* + \cdots + W_n^* \geqq -\log \xi\}, \end{aligned} \tag{5.11}$$

where the W_k^* are independent, and identically distributed with probabilities $P\{W_k^* = \log 2p\} = p$ and $P\{W_k^* = \log 2q\} = q$. Again using the central limit theorem, the problem reduces to showing that

$$\lim_n \frac{\log \xi_{n,t} + nEW_1^*}{\sqrt{n}\, \sigma(W_1^*)} = t \tag{5.12}$$

uniformly in any bounded interval. By a theorem on tail estimates [8], if $X_1, X_2, \cdots$ are independent random variables with $P\{X_k = 1\} = 1/2$ and $P\{X_k = 0\} = 1/2$, then

$$\log P\{X_1 + \cdots + X_n \geqq na\} = n\theta(a) + \mu(n, a) \log n, \tag{5.13}$$

where $\mu(n, a)$ is bounded for all n, with $1/2 + \delta \leqq a \leqq 1 - \delta$, and $\theta(a) = -a \log (2a) - (1 - a) \log [2(1 - a)]$. Now

$$\log \xi_{n,t} = \log [P\{X_1 + \cdots + X_n \geqq np - t\sigma\sqrt{n}\} + 2^{-n}] \tag{5.14}$$

so that the appropriate $a = p - t\sigma/\sqrt{n}$ with

$$\theta(a) = \theta(p) - \frac{t\sigma}{\sqrt{n}} \log \frac{q}{p} + O\left(\frac{1}{n}\right). \tag{5.15}$$

Since $\theta(p) > -\log 2$, we may ignore the 2^{-n} term and estimate

$$\log \xi_{n,t} = n\theta(p) - t\sigma\sqrt{n}\log\frac{q}{p} + O(\log n). \tag{5.16}$$

But $\theta(p) = -EW_1^*$, and the left-hand expression in (5.12) becomes

$$\frac{t\sigma \log \frac{p}{q}}{\sigma(W_1^*)} + O\left(\frac{\log n}{\sqrt{n}}\right). \tag{5.17}$$

Now the short computation resulting, $\sigma(W_1^*) = \sigma \log (p/q)$, completes the proof of the theorem.

There is one final problem we wish to discuss. Fix ξ, with $0 < \xi < 1$, and let

$$T(\xi) = E(\text{first } n \text{ with } S_n \geqq 1 | S_0 = \xi), \tag{5.18}$$

find the strategy which provides a minimum value of $T(\xi)$. We have not been able to solve this problem, but we hopefully conjecture that an optimal strategy is: there is a number ξ_0, with $0 < \xi_0 < 1$, such that if our fortune is less than ξ_0, we use Λ^*, and if our fortune is greater than or equal to ξ_0, we bet to 1, that is, we bet an amount such that, upon winning, our fortune would be unity.

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