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Estimation in Dirichlet Random Effects Models

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Introduction

▶ How this all Started	Prior distributions in the social sciences, survey of political executives
▶ Transition	After the data analysis, look at model properties
▶ Dirichlet Random Effects Model	Likelihood, subclusters, linear and probit models
▶ Precision Parameter	Problems with the MLE and how to fix them
▶ MCMC Sampling Scheme	Parameter expansion, convergence, optimality, generating subclusters, precision parameter
▶ Example and Conclusions	Scottish election data, comparison with normal random effects, future work

———But First———
Here is the Big Picture

- Usual Random Effects Model

$$\mathbf{Y}|\psi \sim N(X\beta + \psi, \sigma^2 I), \quad \psi_i \sim N(0, \tau^2)$$

  - Subject-specific random effect

- Dirichlet Process Random Effects Model

$$\mathbf{Y}|\psi \sim N(X\beta + \psi, \sigma^2 I), \quad \psi_i \sim \mathcal{DP}(m, N(0, \tau^2))$$

- Results in
  - Fewer Assumptions
  - Better Estimates

How This All Started
The Use of Prior Distributions in the Social Sciences

SubsequentQuestions

- When do priors matter in social science research?
- How to specify known prior information?
- Why do Bayesian social scientists tlike uninformed priors?
- Why are reviewers typically skeptical about informed priors, even with lots of supporting evidence?
- Topically:

Can more flexible priors help us recover latent hierarchical information?

Ordinal Model for Survey of Political Executives
Gill and Casella 2008 JASA

- **Outcome Variable:** `stress`
  - ▷ surrogate for self-perceived effectiveness and job-satisfaction
  - ▷ five-point scale from "not stressful at all" to "very stressful."
- Ordered probit model

$$Y_i \sim \text{Multinomial}(1, (p_1, p_2, \dots, p_C)), \quad i = 1, \dots n$$

▷ The p_j are ordered by the probit model

$$p_j = P(\theta_{j-1} \leq U_i \leq \theta_j)$$

where $-\infty = \theta_0 < \theta_1 < \cdots < \theta_C = \infty$.

▷ Defining the random quantity:

$$U_i \sim \mathcal{N}(\mathbf{X}_i \boldsymbol{\beta} + \psi_i, \sigma^2)$$

• $\mathbf{X}_i$: covariates • $\boldsymbol{\beta}$: coefficient vector • ψ_i: subject-specific random effect

Some Distributional Structure

- Freedman (1963), Ferguson (1973, 1974) and Antoniak (1974)
  - ▷ Dirichlet process prior for nonparametric G
  - ▷ Random probability measure on the space of all measures.
- Notation
  - ▷ G_0, a base distribution (finite non-null measure)
  - ▷ $m > 0$, a precision parameter (finite and non-negative scalar)
    - ▷ Gives spread of distributions around G_0,
  - ▷ Prior specification $G \sim \mathcal{DP}(m, G_0) \in \mathcal{P}$.
- For *any* finite partition of the parameter space, $\{B_1, \ldots, B_K\}$,

$$(G(B_1), \ldots, G(B_K)) \sim \mathcal{D}(mG_0(B_1), \ldots, mG_0(B_K)),$$

Ordinal Model for Survey of Political Executives
Specifying the Full Model

- Ordered probit model

$$
\begin{aligned}
Y_i &\sim \text{Multinomial}(1, (p_1, p_2, \ldots, p_C)), \quad i = 1, \ldots n \\
p_j &= P(\theta_{j-1} \leq U_i \leq \theta_j) \\
U_i &\sim \mathcal{N}(\mathbf{X}_i \boldsymbol{\beta} + \psi_i, \sigma^2) \\
\psi_i &\sim G \\
G &\sim \mathcal{DP}(m, G_{\mu,\tau^2}).
\end{aligned}
$$

Model Parameters: $\theta, \mathbf{p}, \beta, \sigma^2$

- ▷ Treated in the usual way
- ▷ Few details here

Dirichlet Parameters: ψ, m

- ▷ Focus of the talk

Survey of Political Executives
Some Coefficient Estimates

Posterior	Mean	95% HD Interval
Government Experience	0.120	[−0.086 : 0.141]
Republican	0.076	[-0.031 : 0.087]
Committee Relationship	-0.181	[-0.302 : -0.168]
Confirmation Preparation	-0.316	[-0.598 : -0.286]
Hours/Week	0.447	[0.351 : 0.457]
President Orientation	-0.338	[-0.621 : -0.309]
Cutpoints: **(None) (Little)**	-1.488	[-1.958 : -1.598]
(Little) (Some)	-0.959	[-1.410 : -1.078]
(Some) (Significant)	-0.325	[-0.786 : 0.454]
(Significant) (Extreme)	0.844	[0.411 : 0.730]

- Intervals are very tight
- Most do not overlap zero
- Seems typical of Dirichlet random effects model (later)
- Reasonable Subject Matter Interpretations

Transition
What Did We Learn?

Analyzing Social Science Data

- ▶ Dirichlet Random Effects Models
  - ▷ Accepted by Social Scientists
  - ▷ Computationally Feasible
  - ▷ Provide good estimates
- ▶ Model fitting was cavalier

Understanding the Methodology

- ▶ "Off the shelf" MCMC ▷ can we do better?
- ▶ Precision parameter m ▷ arbitrarily fixed
- ▶ Answers insensitive to m???

- ▶ Next: Better understanding of MCMC and estimation of m.
- ▶ This is what we really want to talk about!

A Mixed Dirichlet Random Effects Model
Estimating the Dirichlet Parameters

- A general random effects Dirichlet model can be written

$$(Y_1, \ldots, Y_n) \sim f(y_1, \ldots, y_n \mid \theta, \psi_1, \ldots, \psi_n) = \prod_i f(y_i|\theta, \psi_i)$$

  - $\mathcal{DP}$ is the Dirichlet Process with base measure ϕ_0 and precision parameter m.
  - The vector θ contains all model parameters ⇒ Later

- Blackwell and MacQueen (1973) proved that for $\psi_1, \ldots, \psi_n$ iid from $G \sim \mathcal{DP}$,

$$\psi_i|\psi_1, \ldots, \psi_{i-1} \sim \frac{m}{i-1+m}\,\phi_0(\psi_i) + \frac{1}{i-1+m}\sum_{l=1}^{i-1}\delta(\psi_l = \psi_i)$$

  - where δ denotes the Dirac delta function.

A Mixed Dirichlet Random Effects Model
Likelihood Function

- The likelihood function is integrated over the random effects

$$L(\theta \mid \mathbf{y}) = \int f(y_1, \ldots, y_n \mid \theta, \psi_1, \ldots, \psi_n) \pi(\psi_1, \ldots, \psi_n) \, d\psi_1 \cdots d\psi_n$$

$$\pi(\psi_1, \ldots, \psi_n) = \prod_{i=1}^{n} \frac{m\phi_0(\psi_i) + \sum_{j=1}^{i-1} I(\psi_j = \psi_i)}{m + i - 1}.$$

- From Lo (1984 Annals) Lemma 2 and Liu (1996 Annals)

$$L(\theta \mid \mathbf{y}) = \frac{\Gamma(m)}{\Gamma(m+n)} \sum_{k=1}^{n} m^k \left[\sum_{C:|C|=k} \prod_{j=1}^{k} \Gamma(n_j) \int f(\mathbf{y}_{(j)} \,|\theta,\ \psi_j) \phi_0(\psi_j) \, d\psi_j \right],$$

where

- C defines the subclusters
- $\mathbf{y}_{(j)}$ is the vector of y_is in subcluster j
- ψ_j is the common parameter for that subcluster

A Mixed Dirichlet Random Effects Model
Subclusters

A subcluster C is a partition of the sample of size n into k groups

- Grouping done nonparametrically rather than on substantive criteria
- Likely that any real underlying clusters would be broken up into multiple subclusters
- Little penalty for over-separation

- Regular GLMM assumes that the random effect ψ_i's $\sim N(0, \sigma^2_\psi)$, iid.
- Subclustering assigns different ψ_i across groups, common ψ_i within groups.

A Mixed Dirichlet Random Effects Model
Matrix Representation of Subclusters

- Associate a binary matrix $A_{n\times k}$ with of subcluster C

$$C = \{S_1, S_2, S_3\} = \{\{3,4,6\},\{1,2\},\{5\}\} \leftrightarrow A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

  - Rows: a_i is a $1 \times k$ vector of all zeros except for a 1 in its subcluster
  - Columns: The column sums of A are the number of observations in the groups
  - Variables: $\psi_i \in S_j \Rightarrow \psi_i = \eta_j$ (constant in subclusters)

A Mixed Dirichlet Random Effects Model
Underlying Random Effects

$$C = \{S_1, S_2, S_3\} = \{\{3,4,6\},\{1,2\},\{5\}\} \leftrightarrow A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

- Matrix Representation

$$\boldsymbol{\psi} = A\boldsymbol{\eta} \quad \text{where } A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \text{ so } \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_6 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}.$$

- Only need to generate three random variables

A Mixed Dirichlet Random Effects Model
Components of the Likelihood Function

- For each k, $\mathcal{S}_{n,k}$ different subclusters C, the *Stirling Number of the Second Kind*

$$L(\theta \mid \mathbf{y}) = \frac{\Gamma(m)}{\Gamma(m+n)} \sum_{k=1}^{n} m^k \left[\sum_{C:|C|=k} \prod_{j=1}^{k} \Gamma(n_j) \int f(\mathbf{y}_{(j)} \,|\theta,\ \psi_j)\phi_0(\psi_j)\, d\psi_j \right],$$

- $\sum_{k=1}^{n} \mathcal{S}_{n,k} =$ *Bell Number* $= \mathcal{B}_n$
- For example
  - $\mathcal{S}_{25,3} = 141,197,991,025$
  - $\mathcal{B}_{25} = 4,638,590,332,229,999,353$
- Way Cool!

Linear Mixed Model
Effect of the A Matrix

- Start with

$$\mathbf{Y}|\psi \sim N(X\beta + \psi, \sigma^2 I), \text{ where } \psi_i \sim \mathcal{DP}(m, N(0, \tau^2)), \quad i = 1, \ldots, n$$

- Introduce the A matrices to get

$$\mathbf{Y}|\mathbf{A}, \eta \sim N(X\beta + A\eta, \sigma^2 I), \quad \eta \sim N_k(0, \tau^2 I),$$

- Marginalize over η:

$$\mathbf{Y}|\mathbf{A} \sim N(X\beta, \Sigma^*), \qquad \Sigma^* = \left(I + \frac{\tau^2}{\sigma^2} AA'\right)$$

Proof: Fun with Matrix Algebra

- Woodbury's Formula: Inverses
- Sylvester's Theorem: Determinants

Probit Mixed Model
Straightforward Extension

- $Y_i \sim$ Bernoulli$(p_i), \quad i = 1, \ldots, n$
- $g(p_i) = X_i\beta + \psi_i$
- Define the Latent Variable

$$U_i \sim N(X_i\beta + \psi_i, \sigma^2),$$

$Y_i = I(U_i > 0)$
results in probit model

- Adds another step to the Gibbs sampler
- Generate U_i conditional on the other parameters
- Truncated normal random variable generation

- Example Later

Estimating the Precision Parameter
Maximum Likelihood Estimates

- The likelihood function;

$$L(\theta \mid \mathbf{y}) = \frac{\Gamma(m)}{\Gamma(m+n)} \sum_{k=1}^{n} m^k \mathcal{L}_k(\theta|\mathbf{y}),$$

where

$$\mathcal{L}_k(\theta|\mathbf{y}) = \sum_{A \in \mathcal{A}_k} \prod_{j=1}^{k} \Gamma(n_j) \int f(\mathbf{y}|\theta, A)\phi_0(\eta_j)\, d\eta_j.$$

- The MLE $\hat{m}$ solves

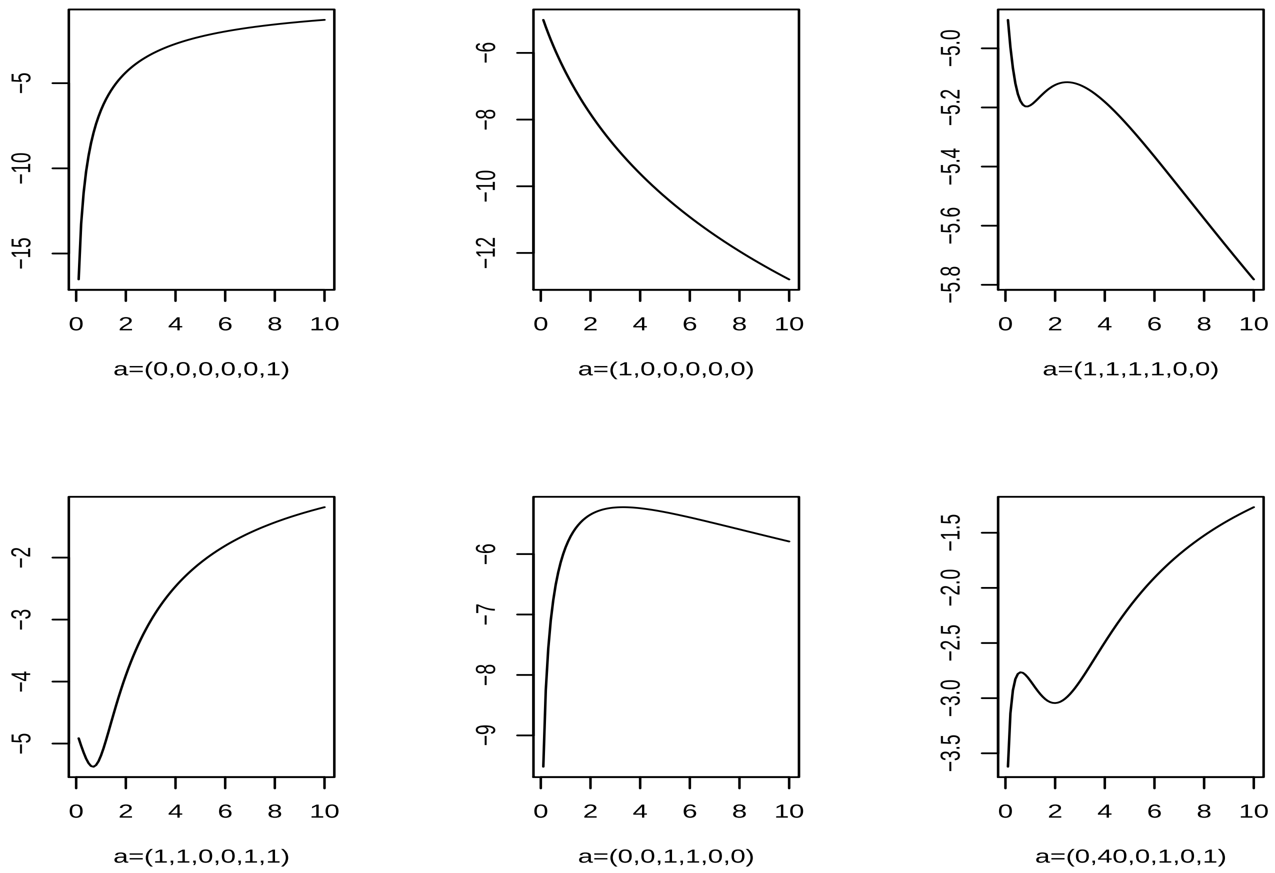
$$\frac{\sum_{k=1}^{n} k m^k \mathcal{L}_k(\theta|\mathbf{y})}{\sum_{k=1}^{n} m^k \mathcal{L}_k(\theta|\mathbf{y})} = \sum_{i=1}^{n} \frac{m}{m+i-1},$$

▷ Equates posterior and prior mean number of clusters (Liu 1996)

- Many surprises here!

Estimating the Precision Parameter
Shapes of the Likelihood Function

- Log likelihoods for a selection of configurations of component likelihoods



- Not Pleasant

Estimating the Precision Parameter
Posterior Mode Estimates

- Given the potential problems with using and MLE for m
- Prior input can be helpful here

Gamma priors previously used

- ▷ Escobar and West (1995)
- ▷ Teh *et al.* (2006)

- Theorem:
  Gamma prior results in

  $\pi(m \mid \mathbf{y}) \uparrow$ from 0

  $\pi(m \mid \mathbf{y}) \downarrow$ to ∞

- Can force interior modal estimate
- Prior can reflect expected number of clusters
- $\widehat{m}$ from $\frac{\partial}{\partial m} \log \pi(m \mid \mathbf{y}) = 0$

MCMC Sampling Scheme

Posterior Distribution

- The joint posterior distribution

$$\pi(\theta, A \mid \boldsymbol{y}) = \frac{m^k f(\boldsymbol{y}|\theta, A)\pi(\theta)}{\int_\Theta \sum_A m^k f(\boldsymbol{y}|\theta, A)\pi(\theta)\, d\theta}.$$

Model	Random effects
Model parameters θ $\longrightarrow$ sampling is straightforward	Dirichlet parameters A : the subclusters m : the precision parameter

MCMC Sampling Scheme
Model Parameters and Dirichlet Parameters

- For $t = 1, \dots T$, at iteration t

Model Parameters

- Starting from $(\theta^{(t)}, A^{(t)})$,

$$\theta^{(t+1)} \sim \pi(\theta \mid A^{(t)}, \boldsymbol{y}),$$

Dirichlet Parameters

- Given $\theta^{(t+1)}$,

$$\mathbf{q}^{(t+1)} \sim \text{Dirichlet}(n_1^{(t)} + \beta_1, \dots, n_k^{(t)} + \beta_k, \beta_{k+1}, \dots, \beta_n)$$

$$A^{(t+1)} \propto m^k f(\boldsymbol{y}|\theta^{(t+1)}, A) \binom{n}{n_1 \; \cdots \; n_n} \prod_{j=1}^{n} [q_j^{(t+1)}]^{n_j}$$

- where $n_j \geq 0$, $n_1 + \cdots + n_n = n$.

MCMC Sampling Scheme – Generating the Subclusters

- For Illustration take $n = 7$. At iteration t

$$A^{(t)}_{7\times 3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Rightarrow \quad q^{(t+1)} | A^{(t)}_{7\times 3} \sim \text{Dirichlet}(4, 3, 3, 1, 1, 1, 1)$$

$$q^{(t+1)} \Rightarrow A^{(t+1)}_{7\times 7} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad A^{(t+1)}_{7\times 5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

MCMC Sampling Scheme – Marginalization

$$q^{(t+1)} \Rightarrow A^{(t+1)}_{7\times 7} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad A^{(t+1)}_{7\times 5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Multinomial Marginalizes Correctly

- No Matrices Needed
- $A^{(t+1)}_{7\times 7}$ represented by $(2, 3, 6, 2, 1, 7, 1)$

MCMC Sampling Scheme
The Transition Kernel

- The joint posterior distribution

$$\pi(\theta, A \mid \boldsymbol{y}) = \frac{m^k f(\boldsymbol{y}|\theta, A)\pi(\theta)}{\int_\Theta \sum_A m^k f(\boldsymbol{y}|\theta, A)\pi(\theta); d\theta}.$$

- The transition kernel of this Markov chain is

  ▷ $P(A|q,\theta) \propto m^k f(\boldsymbol{y}|\theta, A)\binom{n}{n_1 \cdots n_n}\prod_{j=1}^{n} q_j^{n_j}$

  ▷ $q|A \sim \text{Dirichlet}(n_1+\beta_1, \ldots, n_k+\beta_k, \beta_{k+1}, \ldots, \beta_n)$

- $\beta_j = 1$ for all $j \Rightarrow \pi(\theta, A|\boldsymbol{y})$ stationary distribution

MCMC Sampling Scheme
Convergence Properties

- Convergence of Dirichlet :
  - Ignoring the model parameters
  - Concern with convergence of the chain to the stationary distribution

$$\pi(A) = \pi(n_1, \ldots, n_k) = \frac{\Gamma(n)}{\Gamma(n+m)} m^k \prod_{j=1}^{k} \Gamma(n_j),$$

- Neal (2000) describes 8 algorithms
  - Some developed by others
  - All use stick-breaking conditionals

MCMC Sampling Scheme
Parameter Expansion

- The full conditionals of the two chains

Our chain

$$P(a_j = 1|A_{-j}) \propto \begin{cases} \left(\frac{n_j}{n-1+m}\right) \left(\frac{q_j}{n_j+1}\right) & j = 1, \ldots, k \\ \frac{m}{n-1+m} q_{k+1} & j = k+1, \ldots, n \end{cases}$$

Stick-breaking chain

$$P(a_j = 1|A_{-j}) \propto \begin{cases} \frac{n_j}{n-1+m} & j = 1, \ldots, k \\ \frac{m}{n-1+m} & j = k+1 \end{cases}$$

- This is a Parameter Expansion

(Liu/Wu 1999 JASA, vanDyk/Meng 2001 JCGS)

MCMC Sampling Scheme
Optimality

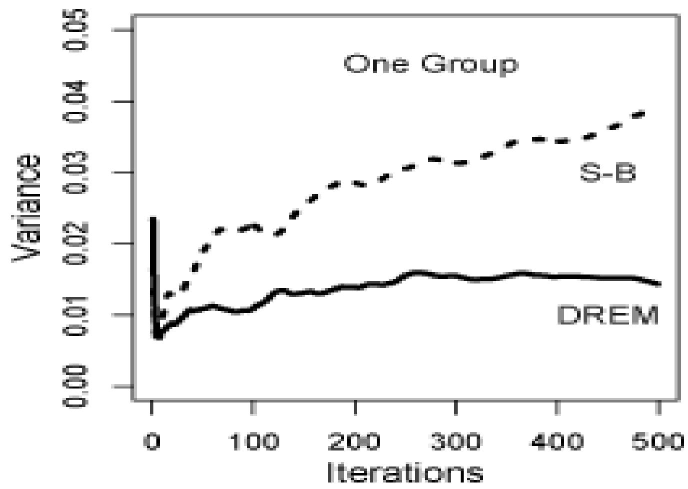
- Hobert and Marchev (2008)
  - ▷ Established superiority of parameter expanded kernels

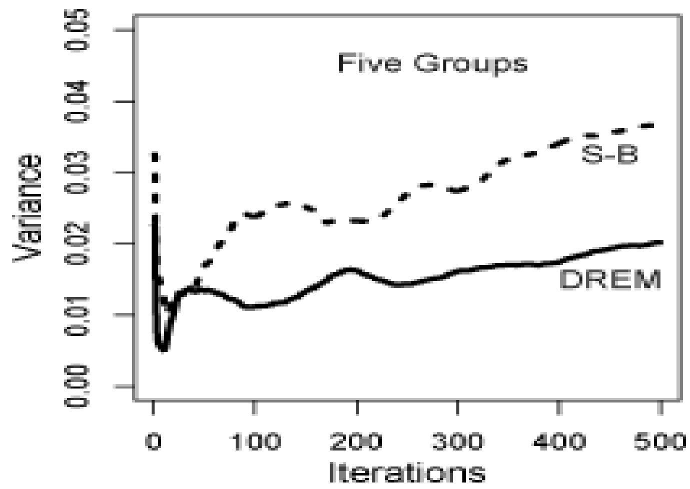
- Properties
  1. Parameter expansion dominates in Operator Norm
  2. Parameter expansion dominates in Efficiency Ordering

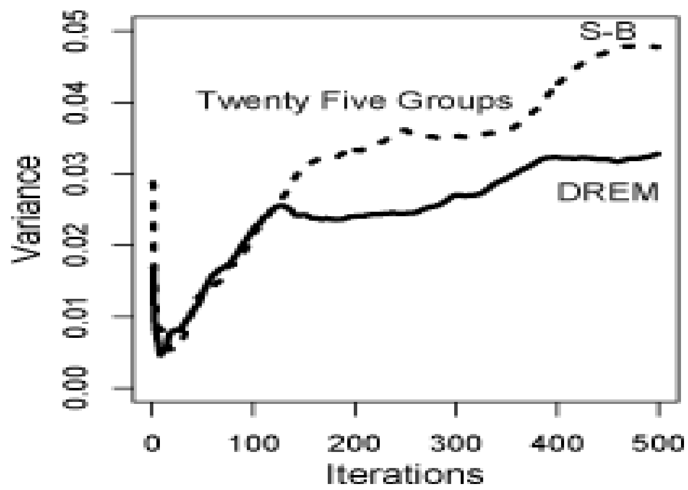
  - Var$h(Y)$ is smaller for any square-integrable function h.

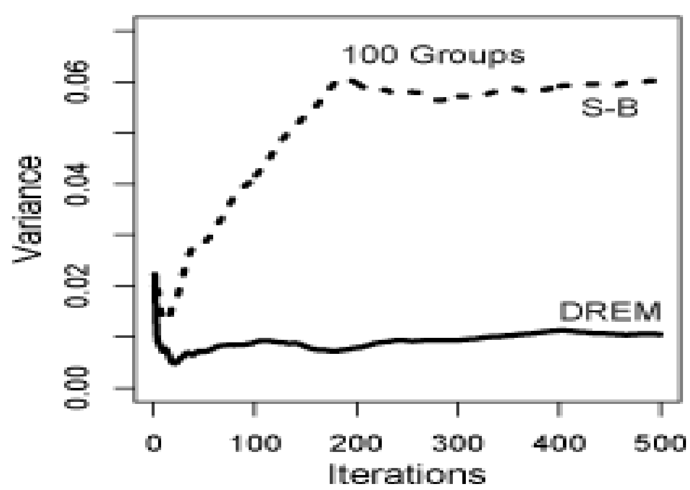
(Mira and Geyer, 1999 and Mira, 2001)

MCMC Sampling Scheme – Variance Comparisons

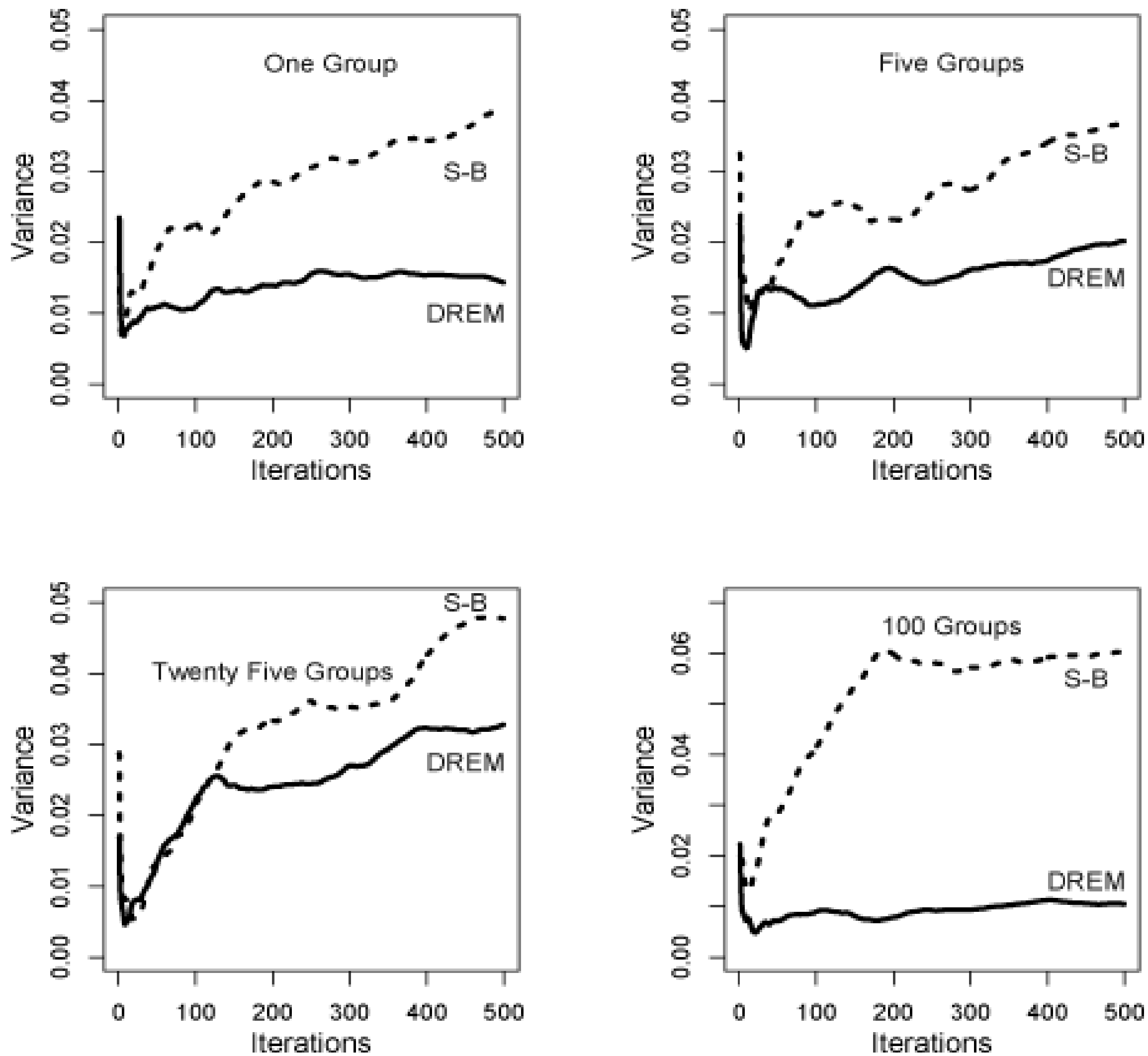








MCMC Sampling Scheme – Variance Comparisons-2



- ▷ $n = 100$
- ▷ 20 Markov chains
- ▷ $1, 5, 25, 100$ true groups
- ▷ Variance at each iteration
- ▷ Improvement $\approx 50\%$

Scottish Election Data - History

1997: Scottish voters overwhelmingly (74.3%) approved the creation of the first Scottish parliament

The voters gave strong support, (63.5%), to granting this parliament taxation powers

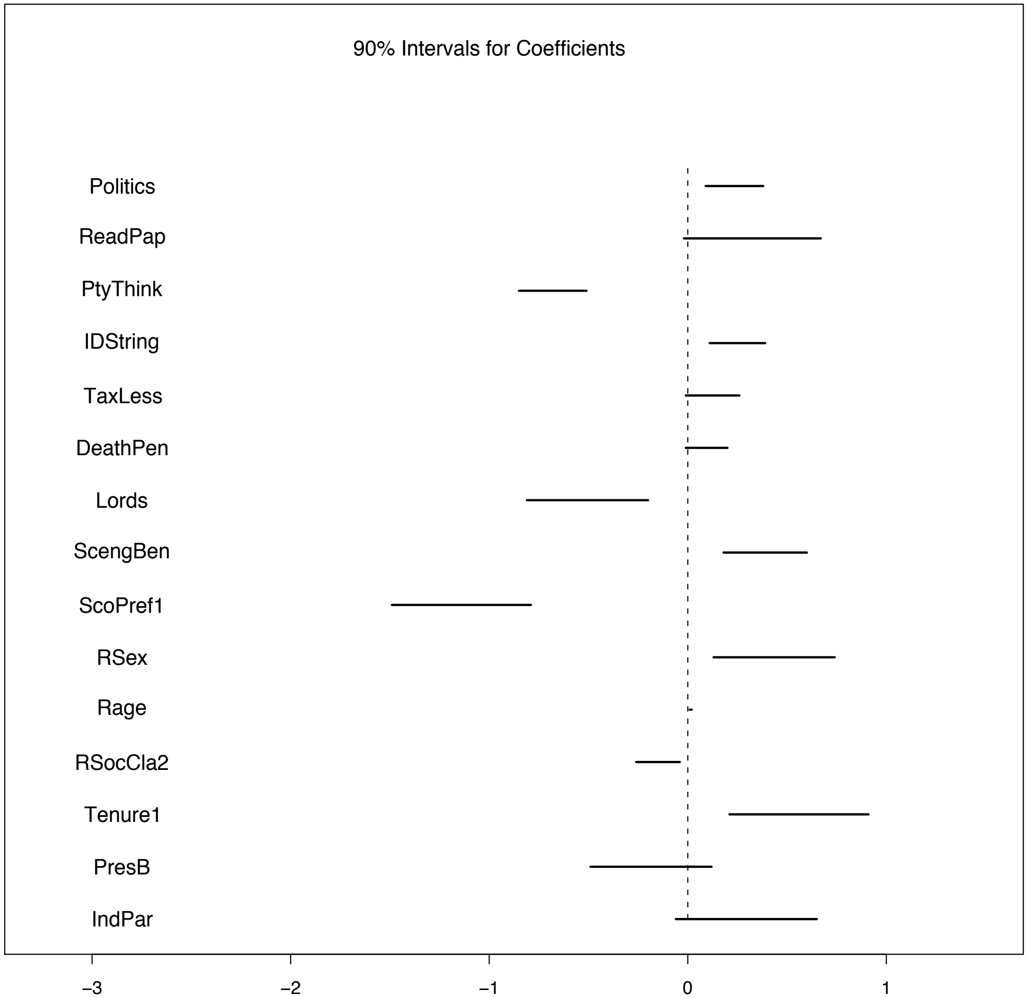
Our Interest:

- Who subsequently voted conservative in Scotland?

The Data:

- British General Election Study of 880 Scottish nationals
- Outcome: party choice (conservative or not) in UK general election
- Independent variables: political and social measures
- Probit model

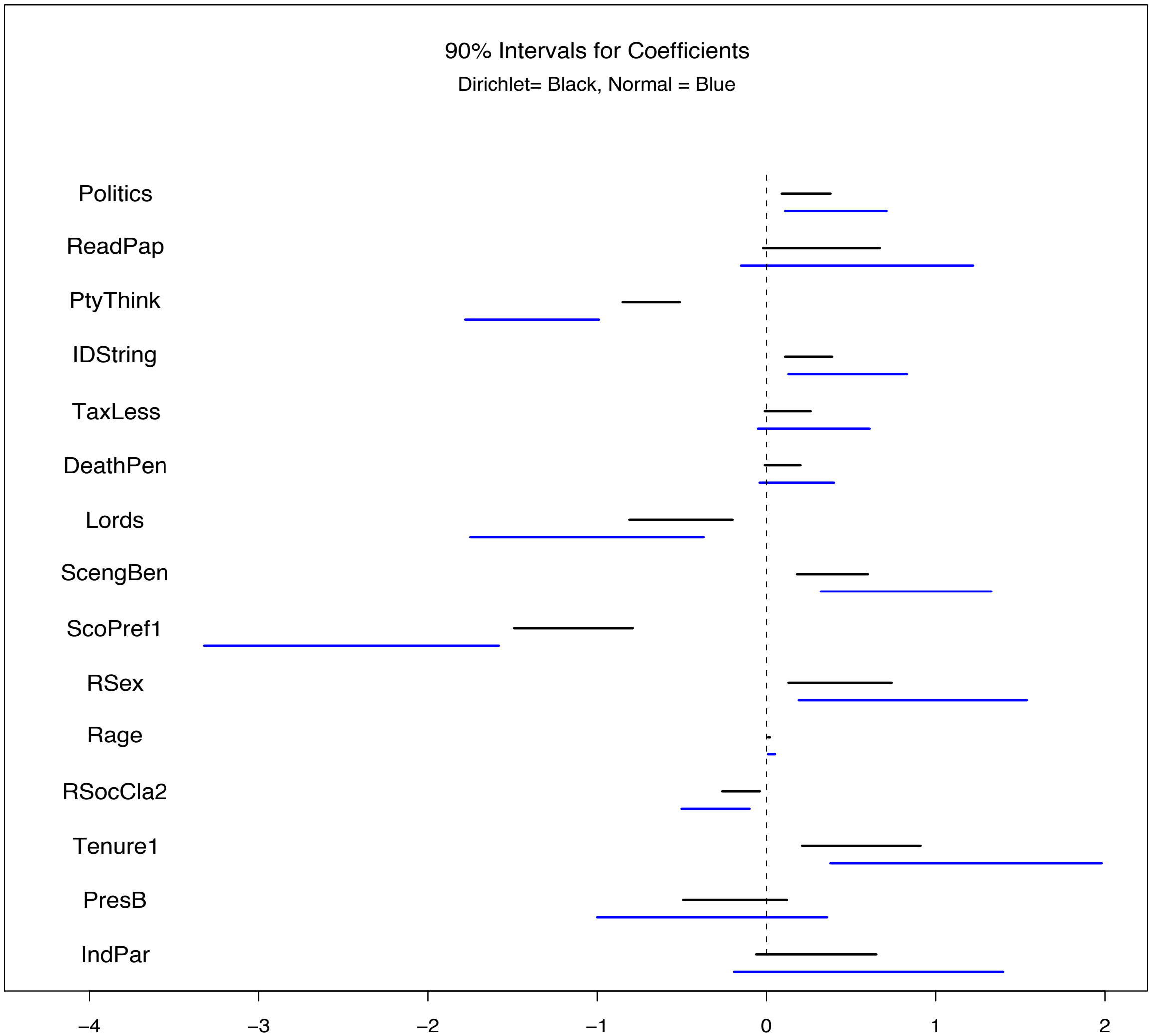
Scottish Election Data - Dirichlet Confidence Intervals



Probability of Voting Conservative ↑ with:

- ▷ Interest in politics (`Politics`)
- ▷ Read newspapers (`ReadPap`)
- ▷ Supports fewer taxes (`TaxLess`)
- ▷ Return death penalty (`DeathPen`)
- ▶ Some Other Surprising Results .....

Scottish Election Data - Confidence Interval Comparison



Dirichlet vs. Normal Random Effects

Dirichlet Intervals Uniformly Shorter

Conclusions

Modelling the Random Effects

Why is the Dirichlet a better model for random effects?

- Balances the data/prior information
- Burr and Doss (2005): No "residual check" for random effects
  - Normality is unverifiable
- Subclustering captures extra variation
  - Shorter Confidence Intervals

Conclusions
Estimation and MCMC

Improvements to the estimation procedure and the MCMC

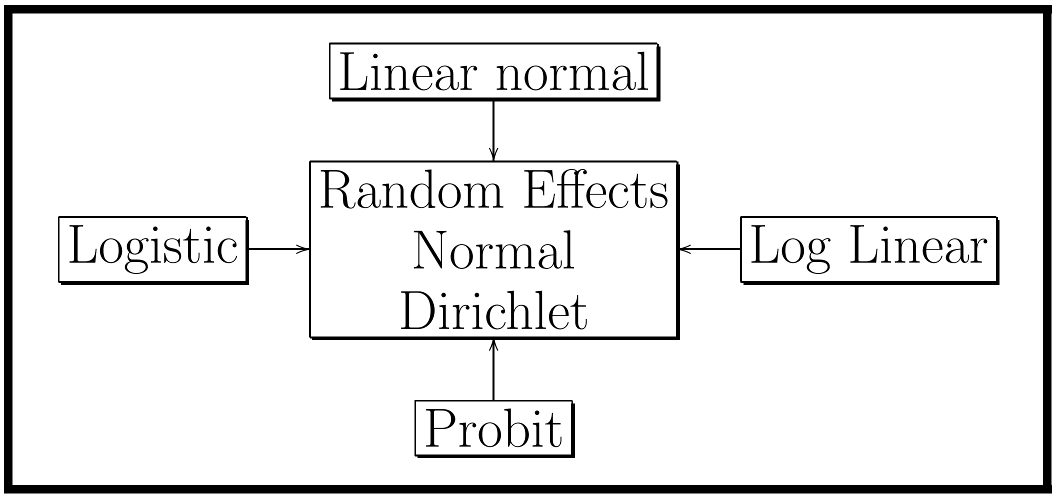
- Matrix representation
  - Allows simplification
- Better precision parameter estimation
- Improved Gibbs sampler
  - Exploits properties of multinomial
  - Better mixing

Conclusions

What Comes Next

Other Models,
Other Methods

- Extend to other models
  - Logit, Loglinear
- More on estimation of m
  - Add to Gibbs Sampler



- Can Mix and Match

Thank You for Your Attention

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