SEQUENTIAL BAHADUR EFFICIENCY

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The notion of Bahadur efficiency for test statistics is extended to the sequential case and illustrated in the specific context of testing one-sided hypotheses about a normal mean. An analog of Bahadur's theorem on the asymptotic optimality of the likelihood ratio statistic is seen to hold in the normal case. Some possible definitions of attained level for a sequential experiment are considered.

1. Introduction. Let $X_1, X_2, \ldots$ be a data sequence of i.i.d. (abstract) random variables whose distributions are indexed by a parameter $\theta$, ranging in a parameter space $\Theta$. We are interested in sequential tests of $H_0: \theta \in \Theta_0$ vs. $H_1: \theta \in \Theta_1 = \Theta - \Theta_0$, where $\Theta_0$ is a nonempty proper subset of $\Theta$. We shall be particularly concerned with the one-sided testing problem for a normal mean, by which is meant that under $\theta$, the data are i.i.d. $N(\theta, 1)$ random variables, $\Theta = (-\infty, \infty)$ and $\Theta_0 = (-\infty, 0]$. Below, we suggest an extension to the sequential case of Bahadur’s theory of the stochastic comparison of tests (more precisely, of test statistics) and of Bahadur efficiency. To that end, a brief synopsis of the non-sequential theory is perhaps in order (cf. Bahadur (1971)).

Let $P_\theta$ denote the (joint) distribution of the data sequence under $\theta$ and let $P_\theta^n$ denote the marginal distribution of $(X_1, \ldots, X_n)$. If, for $\theta, \omega \in \Theta$, we have $P_\theta^n \prec P_\omega^n$, we let $l_n(\theta; \omega) = l(X_1; \theta; \omega) = \log (dP_\theta^n/dP_\omega^n)$ denote the indicated log-likelihood ratio statistic for $X_1$. Similarly, $l_n(\theta; \omega) = \sum_{j=1}^n l(X_j; \theta; \omega) = \log (dP_\theta^n/dP_\omega^n)$. We define the Kullback–Leibler information number

\begin{equation}
K(\theta; \omega) = \int l(\theta; \omega) \, dP_\theta
\end{equation}

if $P_\theta^n \prec P_\omega^n$, otherwise $K(\theta; \omega) = \infty$. We also let

\begin{equation}
K(\theta; \Theta_0) = \inf \{ K(\theta; \omega) : \omega \in \Theta_0 \}.
\end{equation}

The stochastic comparison of tests applies to a sequence $\{T_n\}$ of real-valued test statistics (for testing $H_0$ vs. $H_1$), where $T_n$ is $\mathcal{B}(X_1, \ldots, X_n)$ measurable and large values of $T_n$ are significant. The attained level of $T_n$ is then defined to be $L_n = G_n(T_n)$, where $G_n(x) = \sup \{ P_\omega(T_\omega \geq x) : \omega \in \Theta_0 \}$. We note that $G_n(\cdot)$ is always nonincreasing, hence $L_n$ is measurable. The limit (in probability, if it exists) $r(\theta) = P_\theta - \lim_n [-\log L_n]/n$ will be called the Bahadur index, or simply the index, of the sequence $\{T_n\}$ at $\theta$. (Apart from a factor of 2, this is Bahadur’s

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exact slope.) The index provides a measure of efficiency in that if another sequence \( \{T_n^*\} \) has index \( r^*(\theta) \), to obtain a similar behavior of the attained level with \( n' \) observations (i.e., \( L_{n'} \sim L_n \)), one must have \( n'r^*(\theta) \sim nr(\theta) \). Thus \( r(\theta)/r^*(\theta) \) is, in this sense, the asymptotic relative efficiency of \( \{T_n\} \) to \( \{T_n^*\} \); in short, the Bahadur efficiency. Clearly, large indices are desirable. It follows from a theorem of Raghavachari (1970) (see also Bahadur (1971), Theorem 7.5) that for any such sequence \( \{T_n\} \), one has

\[
P_\theta - \limsup_n \frac{-\log L_n}{n} \leq K(\theta; \Theta_0).
\]

(For a sequence of statistics \( \{Y_n\} \), we write \( P_\theta - \limsup_n Y_n \leq Y \) to mean \( \lim_n P_\theta(Y_n > Y + \delta) = 0 \) for all \( \delta > 0 \). Clearly \( P_\theta - \liminf Y_n = Y \) iff \( P_\theta - \limsup_n Y_n \leq Y \) and \( P_\theta - \limsup_n Y_n \geq Y \).) A sequence \( \{T_n\} \) for which equality holds in (1.3), for all \( \theta \in \Theta_1 \), is called optimal. Under general conditions, Bahadur (1967a, 1971) has shown that the likelihood ratio statistic (sequence)

\[
\Lambda_n = \sup_{\theta \in \Theta_1} \inf_{\omega \in \Theta_0} I_n(\theta; \omega)
\]

is optimal. For the one-sided normal testing problem, \( K(\theta; \Theta_0) = \frac{1}{2} \theta^2 \) and \( \Lambda_n \) is a strictly monotone function of \( S_n/n^2 \), \( S_n = X_1 + \cdots + X_n \), which is optimal in this case.

In a sequential context, we retain a statistic sequence \( \{T_n\} \) and adjoin to this a (randomized) stopping time \( N \) for the data sequence. Unless otherwise noted, we assume \( p(N < \infty) = 1 \) for all \( \theta \in \Theta \). \( T_N \) then denotes the stopped value of the sequence \( \{T_n\} \). Assuming still that large values are significant, we define the attained level of \( T_N \) to be

\[
L_N = H(T_N), \quad \text{where} \quad H(x) = \sup \{P_\theta(T_N \geq x) : \omega \in \Theta_0\}.
\]

To allow for asymptotic considerations, we introduce a family of test sequences and stopping times, \( \{(T(n, a), N_n)\} \), indexed by the real parameter \( a \). Here, for each \( a \), \( T(n, a), n = 1, 2, \ldots \) is a sequence of test statistics and \( N_n \) is a stopping time. We require that for all \( \theta \in \Theta \), \( P_\theta - \limsup N_n = \infty \). (Throughout, limits on \( a \) are taken as \( a \to \infty \).) In many examples, \( a \) is a parameter of the stopping boundary defining \( N_n \) and the boundary moves out as \( a \) increases. Also, in many examples, \( T(n, a) \) does not depend on \( a \). We write \( T(a) = T(N_n, a) \) and denote the corresponding attained level by \( L(a) \). The Bahadur index of \( T(a) \) at \( \theta \) is then defined to be \( P_\theta - \lim \frac{-\log L(a)}{N_n} \), provided this limit exists.

In the sequel, we show that (1.3) remains true in the sequential case (Theorems 2.1 and 2.2). Moreover, for the normal testing problem, we exhibit (Section 3) a sequence \( \{T_n\} \) which is optimal for all families of stopping times. (The statistic is a modification of the likelihood ratio statistic \( S_n/n^2 \).) We also indicate connections between our results and some putative definitions of attained level for a sequential experiment in Section 4. Connections with power curves are discussed in Section 5.

2. Bounds for the Bahadur index. In this section, we give two extensions of
Raghavachari’s theorem to the sequential case. Theorem 2.1 requires only that \( P_\theta - \lim_n N_n = \infty \) and provides a bound for limits in probability of \( \lceil -\log L(a) \rceil / N_n \). With a condition somewhat stronger than \( P_\theta(N_n \to \infty) = 1 \), Theorem 2.2 provides an analogous pointwise result. It is shown by example that the hypothesis of Theorem 2.2 cannot be weakened to \( P_\theta(N_n \to \infty) = 1 \).

**Theorem 2.1.** Let \( L(a) \) be the attained level of \( T(a) \), the stopped value of \( \{T(n, a)\} \) for the stopping time \( N_n \). Then, provided \( P_\theta - \lim_n N_n = \infty \),

\[
(2.1) \quad P_\theta - \limsup_n \lceil -\log L(a) \rceil / N_n \leq K(\theta; \Theta_0).
\]

**Proof.** We suppose first that \( \Theta_0 \) is simple, say \( \Theta_0 = \{\omega\} \), so that \( P_\omega \) denotes the null distribution of the data sequence. We fix \( \theta \in \Theta \) and write \( K = K(\theta; \omega) \). We may suppose \( K < \infty \), since otherwise (2.1) is trivially true. We then write \( l_n(\theta) = l_n(\theta: \omega) \), etc. We also drop the affix \( a \), writing \( N = N_n \), etc. Let \( H(x) = P_\omega(T \geq x) = \sum_n P_\omega(N = n, T_n \geq x) \). Thus \( L = H(T) \). Let \( B = (l_n(\theta) < N(K + \delta)) \). By the SLLN, \( P_\omega(l_n \to K = 1) \) and hence \( P_\theta - \lim_n l_n / N = K \). Thus \( \lim_n P_\theta B = 1 \) for \( \delta > 0 \). Letting \( B^c \) denote the complement of \( B \), for a fixed positive integer \( k \) and arbitrary \( \delta > 0 \), we have

\[
(2.2) \quad P_\omega(L < e^{-n(K+\delta)}) \leq P_\omega(N < k) + P_\omega B^c + P_\omega D_k,
\]

where \( D_k = (N \geq k, L < e^{-n(K+\delta)}, B) \). We note that for \( n \geq k \), \( E_n = (D_k, N = n) = (N = n, H(T_n) < e^{-n(K+\delta)}, l_n < n(K + \delta)) \). We obtain a bound for \( P_\omega E_n \).

\[
(2.3) \quad P_\omega E_n = \int_{X_n} \exp\{l_n(\theta: \omega)\} dP_\omega \\
\leq \exp(n(K + \delta)) P_\omega E_n \leq \exp(n(K + \delta)) P_\omega(L < e^{-n(K+\delta)}) \leq e^{-n\delta}.
\]

the last inequality follows from a result of Bahadur (1971, Theorem 7.4) which says that under any null distribution, an attained level statistic is superuniform on \([0, 1]\). On summing (2.3) over \( n \geq k \), we obtain

\[
(2.4) \quad P_\theta D_k = \sum_{n \geq k} P_\theta E_n \leq e^{-k\delta} / (1 - e^{-\delta}).
\]

We then see from (2.2) that

\[
(2.5) \quad P_\omega(L < e^{-n(K+\delta)}) \leq P_\omega(N < k) + P_\omega B^c + e^{-k\delta} / (1 - e^{-\delta}).
\]

Letting \( a \to \infty \) and then \( k \to \infty \) and recalling that \( P_\theta - \lim_n N = \infty \) and \( \lim_n P_\theta B^c = 0 \), it follows that \( \lim_n P_\theta([ -\log L] / N > K + 2\delta) = 0 \) for any \( \delta > 0 \), which entails (2.1). The case of composite \( \Theta_0 \) follows immediately since \( L \geq L(a, \omega) \) for \( \omega \in \Theta_0 \), where \( L(a, \omega) \) is the attained level of \( T \) for the simple null hypothesis \( \{\omega\} \). Thus we have

\[
P_\theta - \limsup_n [ -\log L] / N \leq P_\theta - \limsup_n [ -\log L(a, \omega)] / N \leq K(\theta, \omega)
\]

for all \( \omega \in \Theta_0 \), which entails (2.1). \( \square \)

Raghavachari’s theorem is actually a pointwise result for \([ -\log L_n] / n\). A corresponding sequential version is given in the next theorem. The framework is
the same as for Theorem 1, except that \( \{N_a\} \) is replaced by a sequence of stopping times.

**Theorem 2.2.** Let \( N_k, k = 1, 2, \ldots \) be a sequence of stopping times for which the following hold: there is a sequence of positive integers \( n_k = n_k(\theta) \) such that

\[
P_\theta \bigcup_{k \geq m} (N_k < n_k) \to 0 \quad \text{as} \quad m \to \infty \quad \text{and} \quad \Sigma_{k} e^{-\delta n_k} < \infty \quad \text{for all} \quad \delta > 0.
\]

Then, letting \( L(k) \) denote the attained level for \( T(k) = T(N_k, k) \),

\[
P_\theta(\lim \sup_k \frac{-\log L(k)}{N_k} \leq K(\theta; \Theta_0)) = 1.
\]

**Remark.** (2.7) entails that \( n_k \to \infty \), hence (2.6) is somewhat stronger than the requirement that \( P_\theta(N_k \to \infty) = 1 \). In fact, (2.6) \( \Rightarrow \) \( P_\theta(\lim \inf_k N_k/n_k \geq 1) = 1 \Rightarrow P_\theta(N_k < n_k(1 - \varepsilon) \ i.o.) = 0 \) for any \( \varepsilon > 0 \). Hence (2.6) is equivalent to the existence of \( \{n_k\} \) for which \( P_\theta(\lim \inf_k N_k/m_k \geq 1) = 1 \). (In many examples, one has the stronger condition that \( P_\theta(N_k/n_k \to 1) = 1 \).) One sees that (2.7) holds if for \( i \neq k, n_i \neq n_k \) and in particular, if \( n_k < n_{k+1} \) for all \( k \). The hypothesis thus holds for the nonsequential case treated by Raghavachari: \( N_k = k \). The hypothesis also holds if \( N_k < N_{k+1}[P_\varepsilon] \) for all \( k \).

**Proof of Theorem.** As in the proof of Theorem 2.1, we again assume first that \( \Theta_0 \) is simple and that \( K = K(\theta; \Theta_0) < \infty \). Let \( B_m = \bigcap_{k \geq m} (l_{N_k} < (K + \delta)N_k) \). Since \( P_\theta(N_k \to \infty) = 1 \), \( \lim_m P_\theta B_m^c = 0 \) by the SLLN. Then

\[
P_\theta \bigcup_{k \geq m} (L(k) < e^{-(K+2\delta)/N_k}) \leq P_\theta \bigcup_{k \geq m} (N_k < n_k) + P_\theta B_m^c + \sum_{k \geq m} P_\theta D_k,
\]

where \( D_k = (N_k \geq n_k, L(k) < e^{-(K+2\delta)/N_k}, B_m) \). As in Theorem 2.1 (cf. (2.4)), it follows that \( P_\theta D_k \leq e^{-\delta n_k}/(1 - e^{-\varepsilon}) \), hence

\[
P_\theta \bigcup_{k \geq m} (L(k) < e^{-(K+2\delta)/N_k}) \leq P_\theta \bigcup_{k \geq m} (N_k < n_k) + P_\theta B_m^c + (1 - e^{-\varepsilon})^{-1} \sum_{k \geq m} e^{-\delta n_k}.
\]

On letting \( m \to \infty \), it follows from the hypothesis that the RHS of (2.10) tends to zero. Thus for any \( \delta > 0 \), \( P_\theta([\lim -\log L(k)]/N_k > K + 2\delta \ i.o.) = 0 \), which entails (2.8) for \( \Theta_0 \) simple. As in Theorem 1, the result for \( \Theta_0 \) composite then follows directly. \( \square \)

An examination of the proof of Theorem 2.2 shows that it utilizes in an essential way the discreteness of the indexing parameter \( k \). It is not clear if there is a satisfactory analog of Theorem 2.2 for a generally indexed collection \( \{N_a\} \). This suggests that it is more appropriate to define indices via probability limits rather than pointwise limits; cf. Bahadur (1960a). This contention is reinforced by the following example, which shows that the conclusion of Theorem 2.2 can fail in a spectacular way if conditions (2.6) and (2.7) do not hold. We assume here the framework of the one-sided normal testing problem.
Example. For $n \geq 1$, let $T_{2n} = S_{2n}/(2n)^t$ and $T_{2n+1} = \infty$. We define a sequence of stopping times, $\{N_k\}$: for $n = 1, 2, \ldots$, let $b_n = e^{x^2}$ and let $\{C_k: b_n < k < b_{n+1}\}$ be a partition of $(-\infty, \infty)$, such that $P_\theta(S_{2n} \in C_k) \leq e^{-x^2}$. This can be done since $b_{n+1} - b_n > e^{x^2}$. Then for $b_n < k < b_{n+1}$ and $n \geq 1$, let

$$N = N_k = 2n$$

$$= 2n + 1$$

Clearly $P_\theta(N_k \to \infty) = 1$ for all $\theta \in \Theta$ and

$$T_N = T_{N_k} = S_{2n}/(2n)^t$$

$$= \infty$$

To compute $L(k)$, the attained level of $T_N$, we note that for $b_n < k < b_{n+1}$

$$\sup \{P_\theta(T_N \geq x): \omega \leq 0 \leq \sup \{P_\theta(S_{2n}/(2n)^t \geq x): \omega \leq 0\} + \sup \{P_\theta(S_{2n} \in C_k): \omega \leq 0\} \leq e^{-x^2 + e^{-x^2}} \leq 2e^{-i - 2x^2}\}.$$ Thus for $b_n < k < b_{n+1}$,

$$[-\log L(k)]/N \geq \frac{1}{2}(T_{2n}/N) - n^{-1}.$$ However, $T_N = \infty$ for exactly one $k$ in the range $b_n < k < b_{n+1}$. Hence

$$\sup \{[-\log L(k)]/N: b_n < k < b_{n+1}\} \geq \frac{1}{2}n - n^{-1}$$

(w.p. 1, under any $\theta \in \Theta$), so that

$$P_\theta(\lim \sup_k [-\log L(k)]/N = \infty) = 1,$$

contrary to (2.8).

Note that (2.11) notwithstanding,

$$P_\theta - \lim \sup_k [-\log L(k)]/N = P_\theta - \lim \sup \frac{1}{2}T_{N_k}/N = P_\theta - \lim \sup \frac{1}{2}S_{2n}/N^t = \frac{1}{2}(N_\theta)^t,$$

so that $\{T_N\}$ has (maximal) index $\frac{1}{2}N_\theta$ at $\theta$. It is easily seen from the above that the corresponding limit inf in (2.11) is $\frac{1}{2}N_\theta$ w.p. 1. This example thus displays a kind of super-efficiency. The example rests heavily on the fact that $N_k$ is essentially constant over long intervals of $k$-values, a possibility excluded by (2.6) and (2.7). The use of improper random variables is not essential here. One may, instead, map $(-\infty, \infty)$ into $(0, 1)$ in an order-preserving way and replace $\infty$ by 1 in the above example.

3. An optimal statistic. As cited above, Bahadur (1967a) showed under general conditions that the likelihood ratio statistic is optimal. It might thus be supposed that this result carries over to the sequential case. We show below (Theorem 3.1) for the one-sided normal testing problem that in a sense, this is so. More specifically, we exhibit a sequence $\{T_n\}$ which has the property that for every collection $\{N_n\}$ of stopping times for which $P_\theta - \lim_n N = \infty$, $T_N$ has maximal index $\frac{1}{2}N_\theta$ under $P_\theta$, $\theta > 0$. We also show by examples that none of the three likely candidates $S_n$, the likelihood ratio statistic $S_n/n^t$ and $S_n/n$ are universally optimal in this sense. Counterexamples for $S_n$ and $S_n/n^t$ are discussed after Theorem 3.1 and one for $S_n/n$ is given in Section 4.

The following result applies to the one-sided normal testing problem.

**Theorem 3.1.** Let $T_n = S_n/n^t - c_n$, where, for some $\delta > 0$, $(2(1 + \delta) \log n)^t \leq c_n = o(n^t)$. Then for any collection $\{N_n\}$ of stopping times for which $P_\theta - \lim_n N = \infty$, $T_{N_n}$ has exact slope $\frac{1}{2}N_\theta$ under $P_\theta$, $\theta > 0$. 

PROOF. We drop the affix $a$, writing $N = N_a$, etc. For $x > 0$,
\[
\sup \{P_a(T_N \geq x) : \omega \leq 0\} = \sup \{P_a(S_N/N^t - c_N \geq x) : \omega \leq 0\} \\
\leq \sum_{n=1}^{\infty} \sup \{P_a(S_n/n^t - c_n \geq x) : \omega \leq 0\} \\
= \sum_{n=1}^{\infty} P_a(S_n/n^t \geq x + c_n) \leq \sum_{n=1}^{\infty} \exp\{-\frac{1}{2}(x^2 + c_n^2)\} \\
\leq e^{-\frac{1}{2}\theta^2} \sum_{n} n^{-\frac{1}{(1+\delta)}} = C(\delta)e^{-\frac{1}{2}\theta^2}.
\]

Letting $L$ denote the attained level of $T_N$, it follows that $-\log L \geq \frac{1}{2}(S_N/n^t - c_N)^2 - \log C(\delta)$ on $(S_N > N^t c_N)$, so that $P_\theta - \liminf_{n} \{-\log L)/N \geq P_\theta - \lim_n (\frac{1}{2}(S_N/N - c_N/N^t)^2 - N^{-1}\log C(\delta)) = \frac{1}{2}\theta^2$ since $P_\theta - \lim c_N/N^t = 0$ and $P_\theta(S_N > N^t c_N) \to 1$. By Theorem 2.1, $P_\theta - \limsup_{n} \{-\log L)/N \leq \frac{1}{2}\theta^2$. Thus $T_N$ has (maximal) exact slope $\frac{1}{2}\theta^2$ under $P_\theta$. \qedsymbol

REMARK. The estimate used to establish the theorem is very crude and it is conjectured that $\{T_n\}$ remains optimal for all $c_n = o(n^t)$ for which $n^t c_n$ is in the upper class of LIL (for $S_n$). By a similar argument, it can also be shown that another modification of $S_n/n^t$ that renders it optimal for all stopping times is $(S_n/n^t)/(S_n > n^t c_n)$, where $c_n$ satisfies the conditions given in Theorem 3.1.

We show next by example that $S_n/n^t$ need not be optimal. Heuristically, this may be seen by considering the stopping time $N = \inf \{n : |S_n| \geq (an^t)\}$. Neglecting overshoot, $S_n/N^t$ assumes only the values $-at$ and $at$, having corresponding attained levels 1 and $\frac{1}{2}$. Since $P_\theta - \lim N = \infty$, the exact slope of $S_n/n^t$ is zero. (The argument becomes rigorous if $S_n$ is replaced by a continuous-time Wiener process.) A rigorousization of the above heuristics for discrete time appears to be very delicate. Instead, we consider another stopping boundary for which the analysis is simpler. We introduce the following notation. For $x > 0$, let $e_k(x) = l_k(x) = x$, $l_{k+1}(x) = \max\{\log l_k(x), 1\}$ and $e_{k+1}(x) = \exp(e_k(x))$, $k = 0, 1, \ldots$. Thus $l_k$ and $e_k$ are inverse to each other on $[e_k(1), \infty)$. Let
\[(3.1) \quad N = \inf \{n : S_n \geq (an_k(n))^t\}.
\]

By the SLLN and LIL, $P_{\theta}(N < \infty) = 1$, all $\theta \geq 0$. (It would perhaps appear more natural, in the testing context, to define $N$ by a two-sided boundary, with $|S_n|$ replacing $S_n$ in (3.1). This requires no essential change in the following argument.) We note that a reasonable sequential test can be based on the data sequence stopped by $N$. One may, for example, reject $H_0$ for large values of $S_n/N^t - \log N$ (cf. Theorem 3.1).

PROPOSITION 3.2. With $N$ as in (3.1), $S_n/N^t$ has exact slope zero under all $\theta > 0$.

PROOF. We use the following crude estimate:
\[(3.2) \quad P_\theta(N \leq n) \leq \sum_{k=1}^{n} P_\theta(S_k/k^t \geq (al_k(k))^t) \leq ne^{-n^t}.
\]

We next estimate the tail of the statistic $S_n/(aN)^t$. Since $S_n/(aN)^t \geq [l_k(N)]^t$, we have, for $x > 1$,
\[
\sup \{P_a(S_n/(aN)^t \geq x) : \omega \leq 0\} \geq P_a(S_n/(aN)^t \geq x) \geq P_a(l_k(N) \geq x^2) \\
= P_a(N \geq e_k(x^2)) \geq 1 - e^{-\frac{1}{2}\theta^2} e_k(x^2)
\]
by (3.2). It will follow that \( L \), the attained level of \( S_N/N^\dagger \) satisfies
\[
(3.3) \quad P_\theta(L \to 1) = 1, \quad \theta > 0,
\]
if we show that
\[
(3.4) \quad P_\theta(e^{-i\pi} e_\delta(S_N^\dagger/aN) \to 0) = 1, \quad \theta > 0.
\]

To establish (3.4), we note first that
\[
(3.5) \quad 0 \leq S_N - [aNl_\delta(N)]^\dagger \leq X_N.
\]
It is an easy consequence of the Borel–Cantelli lemmas that for normal variables with unit variance,
\[
(3.6) \quad P_\theta(\limsup_n X_n/(2 \log n)^\dagger = 1) = 1, \quad \text{all } \theta,
\]
hence
\[
X_N = O([\log N])^\dagger.
\]
(For random variables \( Y_a \) and \( Z_a \), we write \( Z_a = O(Y_a) \) to mean \( \limsup_n |Z_n/Y_n| < \infty \) w.p. 1. A corresponding meaning is given to the expression \( Z_a = o(Y_a) \).)

On dividing across by \( N \) in (3.5) and noting that \( P_\theta(S_N/N \to \theta) = 1 \), we see that \( P_\theta(N/\alpha l_\delta(N) \to 1/\theta^2) = 1 \), hence that
\[
(3.8) \quad P_\theta(N/\alpha l_\delta(a) \to 1/\theta^2) = 1, \quad \theta \geq 0.
\]

It follows from (3.5)—(3.8) that
\[
(3.9) \quad S_N^\dagger/(aN)^\dagger = l_\delta(N) + O([l_\delta(a)]^\dagger/a).
\]

On taking exponentials, we see that on \( N > e_\delta(1) \),
\[
e_\delta(S_N^\dagger/aN) = l_\delta(N)[1 + O([l_\delta(a)]^\dagger/a)] = l_\delta(N) + O(l_\delta(a)[l_\delta(a)]^\dagger/a),
\]
hence that
\[
(3.10) \quad e_\delta(S_N^\dagger/aN) = l_\delta(N) + O(l_\delta(a)[l_\delta(a)]^\dagger/a) \quad \text{and}
\]
\[
e_\delta(S_N^\dagger/aN) = N[1 + o(1)].
\]

It follows from (3.10) and (3.8) that (3.4) and hence (3.3) hold. A fortiori, \( S_N/N^\dagger \) has exact slope zero under all \( \theta > 0 \).

The next example shows that \( S_N \) need not be optimal. The no-overshoot heuristics are similar to those for \( S_N/N^\dagger \) above. Let
\[
(3.11) \quad N = N_a = \inf \{ n : |S_n| \geq a \}.
\]

The following result implies that \( S_N \) has index zero. Let \( L(a) \) denote the attained level for \( S_N \).

**Proposition 3.3.** As \( a \to \infty \),
\[
(3.12) \quad \log L(a) = o_\pi(N_a).
\]
Remark. Note that $N_a \to \infty$. In fact, from Theorem 2.1 of Berk (1973), we have for $\theta > 0$ that

$$\lim_{a} N_a/a = 1/\theta \quad [P_\theta].$$

We establish (3.12) with the aid of the following lemmas.

Lemma 3.4. Let $N = N_a$ be as in (3.11). Let $\Phi$ denote the $N(0, 1)$ df. For $x > 0$,

$$P_\theta(S_N \geq a + x) \geq (1 - \Phi(x + o(\log a))]e^{\theta(1)}.$$  

Proof. Fix a positive $k < a$. Then

$$P_\theta(S_N \geq a + x) \geq P_\theta(S_{N-1} \geq a - k, X_N \geq x + k) = \sum_n P_\theta(N \geq n, S_{n-1} \geq a - k, X_n \geq x + k) \geq [1 - \Phi(x + k)] \sum_n P_\theta(N = n, S_{n-1} \geq a - k) \geq [1 - \Phi(x + k)] P_\theta(S_{N-1} \geq a - k) \geq [1 - \Phi(x + k)] P_\theta(X_N \leq k, S_N \geq a) = [1 - \Phi(x + k)] P_\theta(S_N \geq a) - P_\theta(X_N > k).$$

We note next that $P_\theta(X_N > k) \leq E_\theta \sum_i I(X_j > k) = E_\theta NP_\theta(X_i > k) \leq e^{-k^2} E_\theta N$. We have from Theorem 2.4 of Berk (1973) that $E_\theta N = O(a^2)$, hence, taking $k = (6 \log a)^{\frac{1}{2}}, P_\theta(X_N > k) = O(a^{-1})$. The desired conclusion then follows from the fact that $P_\theta(S_N \geq a) = \frac{1}{2}$. 

Lemma 3.5. Let $N$ be as in (3.11). Then for all $\theta$, $E_\theta(S_N - a | S_N \geq a) = O(1)$, hence $(S_N - a)I(S_N \geq a) = O_p(1)$ for all $\theta \geq 0$.

Proof. A familiar argument, due originally to Wald (1947) gives $E_\theta(S_N - a | S_N \geq a) = E_\theta(X_N - (a - S_{N-1}) | X_N \geq a - S_{N-1}) \leq \sup \{E_\theta(X_1 - u | X_1 \geq u) : 0 \leq u \leq a + b\} = E_\theta(X_1 | X_1 \geq 0) = E_\theta(X_1 + \theta | X_1 \geq -\theta) = w(\theta), \text{ say.}$ Clearly $w(\theta) < \infty$ and does not depend on $a$, which entails the desired conclusion.

Remark. For the stopping time $\tau = \tau_a = \inf \{n : S_n \geq a\}$, it follows from known results in renewal theory (see, e.g., Feller (1971), page 371) that $S_\tau - a$ has a proper limit law as $a \to \infty$ and is a fortiori $O_p(1)$. This entails the second part of the lemma since on $(S_N \geq a)$, $S_N = S_\tau$. (In fact, for $\theta > 0$, $S_N - a$ and $S_\tau - a$ have the same limit law for then $\lim_{a} P_\theta(S_N \geq a) = 1$.)

Proof of Proposition 3.3. We obtain (3.12) by noting that $\sup \{P_\theta(S_N \geq a + x) : \omega \leq 0\} \geq P_\theta(S_N \geq a + x)$, so that Lemma 3.4 entails $-\log L(a) \leq \frac{1}{2}[S_N - a + o(\log a)^2] + O(1)$. The desired conclusion then follows from Lemma 3.5 and (3.13).

4. Bahadur efficiency and sequential experiments. We consider here, in the context of the one-sided testing problem for a normal mean, possible definitions of the attained level of a sequential experiment. For us, the term "sequential
experiment” is synonymous with “stopping time.” In speaking of the attained level of an experiment, one has tacitly in mind, of course, an appropriate null hypothesis. We suppose that the stopping time is defined by a well-behaved (say, convex) continuation region in the \((n, S_n)\) plane. A point in this plane is denoted by \((n, s)\). In the nonsequential case, when the stopping time is \(N \equiv n\), the stopping boundary is a vertical line at \(n\) and the second coordinate orders the boundary, providing a measure of significance or attained level for the possible outcomes (with \(s = \infty\) the most significant value and \(s = -\infty\) the least significant). It seems natural to try to similarly order other stopping boundaries and analogously define attained levels for the outcomes of a sequential experiment. For example, consider the triangular (Anderson) boundary corresponding to \(N = \inf\{n : |S_n| \geq a - n\}\), the upper boundary being the segment of the line \(s = a - n\) between \((1, a - 1)\) and \((a, 0)\). Intuitively (neglecting overshoot), the portion of the boundary near \((1, a - 1)\) contains the most significant outcomes (outcomes that speak most against the null hypothesis \(\theta \leq 0\)) while the boundary near \((1, 1 - a)\) contains the least significant outcomes. It seems natural to order the stopping boundary with the outcome \((1, a - 1)\) having the smallest attained level, proceeding clockwise around to the outcome \((1, 1 - a)\), which then has attained level one. The actual attained level of a point \((n, a - n)\) on the upper boundary, say, is the (maximum) null probability of stopping on the portion of the boundary between \((1, a - 1)\) and \((n, a - n)\). In this case, this is just the null probability (neglecting overshoot) that \(S_n/N \geq (a - n)/n\), so that this notion of attained level for the sequential experiment is just the attained level of \(S_n/N\). This reasoning implicitly uses the idea that under \(\theta_1\), \((n, S_n)\) tends to move along the ray \(s = n\theta\), so that one expects large values of \(S_n/N\) under the alternative hypothesis.

These considerations apply as well to other familiar stopping rules defined by an upper and a lower stopping boundary, the SPRT for example. (For open boundaries such as the SPRT or the square-root boundary mentioned in Section 3, the ordering still proceeds clockwise starting from \(n = 1\) on the upper boundary. However, one must make an imaginary transition at \(n = \infty\) to reach the lower boundary.) For these stopping times (still neglecting overshoot), the attained level of the point \((n, s)\) on the upper boundary is also the null probability of exiting at the upper boundary with \(N \leq n\). Thus another possible definition of attained level for the stopping time is the attained level of the statistic \(N\), small values being significant (and subject to the proviso that one exits at the upper boundary). If overshoot is not neglected, the two definitions do not quite coincide, except asymptotically. We indicate below (Theorem 4.1) a class of boundaries for which this notion of attained level is optimal, in that the corresponding exact slope is maximal. Our considerations apply to \(S_n/N\), though there are similar results for \(N\). We also show by example that \(S_n/N\) is not optimal in every case, even for a convex continuation region whose boundary permits a clockwise ordering as discussed above.
The following gives conditions under which \( S_n/N \) is optimal. We consider continuation regions whose upper halves have asymptotic shapes: on homothetically contracting by a factor \( a \), the contracted regions tend to a limit.

**Theorem 4.1.** Let \( N = \inf \{ n : S_n \notin (-a f_a(n/a), a g_a(n/a)) \} \), where the nonnegative boundary curves \( f_a \) and \( g_a \) satisfy the following:

(i) For \( n = 1, 2, \ldots \), \( \lim_n a f_a(n/a) = \infty \) and \( \lim_{x \to \infty} f_a(x)/x = 0 \).

(ii) \( g_a^2(x)/x \) is nonincreasing in \( x \).

(iii) \( R_a(x) = g_a(x)/x \) decreases with \( a \) to a continuous strictly decreasing limit \( R(x) \), where \( R(0+) > 0 \).

Then \( L \), the attained level of \( S_n/N \) satisfies

\[
P_\theta - \lim_n [-\log L]/N = \frac{1}{2} \theta^a, \quad \theta > 0.
\]

**Remark.** Here is an example of a continuation region satisfying the hypothesis of the theorem. For \( b > 0 \) (which will depend on \( a \) in a manner specified below), let \( N = \inf \{ n : |S_n| \geq (bn \log n)^4 \} \). Setting \( n/a = x \) and \( a g_a(x) = (bn \log n)^4 \), \( g_a(x) = (b a x^{-1}[\log a + \log x])^4 \). If we now set \( b = a/\log a \), we see that \( g_a(x) \to x^4 \). Thus the asymptotic shape of a \((n \log n)^4\) boundary is a "root-n" boundary.

**Proof.** If follows from the hypothesis that \( R_a(\cdot) \) decreases continuously to zero. Let \( \eta_a = R_a^{-1} \). Then \( \eta_a \) is increasing and decreases with \( a \) to \( \eta = R^{-1} \). For \( \theta > 0 \), we have

\[
(4.1) \quad \lim_n P_\theta(S_n \geq a g_a(N/a)) = 1,
\]

which, together with the relation \( S_{N-1} < a g_a(N/a - 1/a) \) entails (on dividing across by \( N \)) \( P_\theta(\lim_n R_a(N/a) = \theta) = 1 \). The hypotheses insure that \( R_a \) converges uniformly to \( R \). Thus \( P_\theta(\lim_n R(N/a) = \theta) = 1 \) or

\[
(4.2) \quad P_\theta(N/a \to \eta(\theta)) = 1.
\]

Next we note that for \( x > 0 \), letting \( g(x) = x R(x) \),

\[
\sup \{ P_\theta(S_n \geq N x) : \omega \leq 0 \}
\]

\[
\leq \sum_n P_\theta(S_n \geq n x \vee a g(n/a))
\]

\[
\leq \sum_{n \leq a g(n/a)} P_\theta(S_n \geq a g(n/a)) + \sum_{n > a g(n/a)} P_\theta(S_n \geq n x)
\]

\[
\leq \sum_{n \leq a g(n/a)} \exp\left\{-\frac{1}{2} a^2 g^2(n/a)/n\right\} + \sum_{n > a g(n/a)} \exp\left\{-\frac{1}{2} n x^2\right\}
\]

\[
= \exp\left\{-\frac{1}{2} a g(\gamma(x)) \right\} [\gamma(x) [1 + o(1)]] + \exp\left\{-\frac{1}{2} a \gamma(x) x^2 [1 + o(1)]\right\}
\]

\[
= \exp\left\{-\frac{1}{2} a \gamma(x) x^2 [1 + o(1)]\right\},
\]

since \( g(\gamma(x))/\gamma(x) = x \). It follows that under \( \theta > 0 \), \( L \), the attained level of \( S_n/N \) satisfies

\[
(4.4) \quad -\log L \geq \frac{1}{2} a(S_n/N)^a S_n/N [1 + o(1)].
\]

Since \( P_\theta(S_n/N \to \theta) = 1 \), it follows from (4.2) that \( S_n/N \) has index \( \frac{1}{2} \theta^a \) under \( P_\theta \).
The following example shows that the above intuitive notion of attained level for a sequential statistic is not always optimal. (Alternatively, $S_n/N$ is not always an optimal statistic.) Let $N = \inf \{ n : S_n \geq a_1 n^2 \}$ or $n \geq a_1$. The truncation assures that $P_\theta(N < \infty) = 1$ for all $\theta$. It follows as in Theorem 4.1 that

$$N/a \to 1/\theta^2 = \eta(\theta), \quad \theta > 0.$$  

(Asymptotically, the truncation has no effect for $\theta > 0$.) Moreover, for $x > 0$,

$$\sup \{ P_\theta(S_n \geq N_x : \omega \leq 0) \geq P_\theta(S_n \geq N_x) \geq P_\theta(X_1 \geq a_1) = e^{-x^3} [1 + o(1)].$$

Thus $L$, the attained level of $S_n/N$ satisfies $-\log L = O(a^4)$, which, in view of (4.5) implies that $S_n/N$ has exact slope zero under all $P_\theta$. In the above example, we get the same result for any limiting boundary curve $g(x)$ for which $g'(x)/x$ increases to $\infty$ as $x \uparrow \infty$. It follows from Theorem 3.1 that there are still optimal statistics in such cases: $S_n/N^\frac{1}{4}$, for example. In fact, it may be verified that $S_n/N^\frac{1}{4}$ is optimal here.

Theorem 4.1 suggests that Bahadur efficiency is not helpful for distinguishing among stopping times of interest. If the above intuitive notion of attained level of a sequential experiment is adopted, all of the stopping times subsumed by Theorem 4.1 seem equally efficient. If, instead of $S_n/n$, one uses $T_n = S_n/n^\frac{1}{4} - \log n$ to define the attained level of a sequential experiment, by Theorem 3.1 all stopping times have (absolute and relative) efficiency one. One reaches the same conclusion if the index of a sequential experiment is taken to be the supremum of all indices attained by statistics defined on the stopped data sequence. The relevance of this for sequential testing is not entirely clear. At any rate, the notions of Hodges–Lehmann and Chernoff efficiency do serve to distinguish among sequential tests; see Berk (1976).

5. Some connections with significance levels. The Bahadur index we have been considering is more properly called the stochastic Bahadur index. An allied notion, a nonstochastic Bahadur index, provides a measure of efficiency for critical regions; cf. Bahadur (1960a, b). In the nonsequential case, for critical regions defined by a sequence of test statistics $T_n$, this latter notion is the following: one selects critical values $c_n = c_n(\beta)$ so that $\beta < P_{\theta}(T_n \geq c_n) < 1 - \beta$ for some $\beta \in (0, \frac{1}{2})$ and all $n$. Then, letting $\alpha_n = \alpha_n(\beta) = \sup \{ P_{\theta}(T_n \geq c_n) : \omega \in \Theta_0 \}$ be the associated size, the nonstochastic index for (the critical regions defined by) $T_n$ is

$$\rho(\theta) = \lim_n -\frac{1}{n} \log \alpha_n,$$

provided this limit exists.

Judging from the loose way in which the term "Bahadur efficiency" is used in the literature, the distinction between stochastic and nonstochastic indices is not always uppermost in people's minds. Happily, the two notions often coincide. Indeed, Bahadur (1967b, Proposition 11) has established a result of this nature in the nonsequential case: if $\{T_n\}$ has stochastic index $r(\theta)$ which is
constant w.p. 1, then \( \rho(\theta) = r(\theta) \). Some authors (Klotz (1965), Stone (1968), Tsutakawa (1968)) have computed only nonstochastic indices. In such cases, it would be of interest to know that this is also the stochastic index. One can, of course, verify the existence of a constant stochastic index and appeal to Bahadur's result. This seems tantamount to computing the stochastic index. An alternative is provided by Theorem 5.2 below, which avoids treating stochastic limits.

An appropriate sequential analog of (5.1) is not entirely obvious. Should one replace the divisor \( n \) by \( N \), or, to retain the nonstochastic nature of the index, use \( E_\theta N \) or some other deterministic quantity? Fortunately, in many cases one does not have to confront this difficulty, since there exist constants \( \nu_a = \nu_a(\theta) \) for which

\[
P_\theta - \lim_n N_n/\nu_a = 1.
\]

Usually, one has \( \lim_n E_\theta N_n/\nu_a = 1 \) as well. Assuming the choice of divisor to be thus resolved, Theorems 5.1 and 5.2 below give conditions under which the stochastic and nonstochastic indices coincide. Theorem 5.1 is similar to Bahadur's Proposition 11 (op. cit.), in that one essentially assumes the existence of a stochastic index. Theorem 5.2 is something of a converse proposition, in which one infers the existence of a stochastic index.

**Theorem 5.1.** Let \( T(a) \) be the stopped statistic for \( N_a \) and let \( L(a) \) denote the attained level of \( T(a) \). Suppose that for some \( \beta \in (0, \frac{1}{2}) \), there are critical values \( c_a = c_a(\theta) \) for which

\[
\beta < P_\theta(T(a) \geq c_a) < 1 - \beta, \quad \text{all } a.
\]

Let \( \alpha_a = \alpha_a(\theta) = \sup\{P_\theta(T(a) \geq c_a) : \omega \in \Theta_\theta\} \). Then, if there are constants \( \nu_a = \nu_a(\theta) \) for which

\[
P_\theta - \lim_n [-\log L(a)]/\nu_a = k, \quad k = 1 \text{ or } 0
\]

then also

\[
\lim_n [-\log \alpha_a]/\nu_a = k.
\]

If \( k = 1 \), we can take \( \nu_a = -\log \alpha_a \). The possibility \( k = 0 \) allows for an index of zero. Verifying (5.4) is akin to verifying the existence of a stochastic index. If \( P_\theta - \lim N_a/\nu_a \) exists and is not zero (w.p. 1), then (5.4) is equivalent to the existence of a stochastic index. The next result allows us, in a sense, to reverse the logical implication (5.4) \( \Rightarrow \) (5.5) and deduce the existence of a stochastic index from corresponding behavior of \( \log \alpha_a \).

**Theorem 5.2.** Let \( T(a) \) and \( L(a) \) be as in Theorem 5.1. Suppose there are subsets \( B_a \) and \( B_1 \) of \([0, 1]\) and, for all \( \beta \in B_a \cup B_1 \), critical values \( c_a(\beta) = c_a(\beta, \theta) \) which satisfy:

\[
\inf\{\beta : \beta \in B_a\} = 0, \quad \sup\{\beta : \beta \in B_1\} = 1.
\]

(a) \( \forall \beta \in B_1, \liminf_a P_\theta(T(a) \geq c_a(\beta)) \geq \beta \)

(b) \( \forall \beta \in B_a, \limsup_a P_\theta(T(a) > c_a(\beta)) \leq \beta \).
Let \( \alpha_{\delta}(\beta) = \alpha_{\delta}(\beta, \theta) = \sup \{ P_{\omega}(T(a) \geq c_{\delta}(\beta)) : \omega \in \Theta_{\omega} \} \). Then, if there are constants \( \nu_{\omega} = \nu_{\omega}(\theta) \) for which

\[
\lim_{a} \left[ -\log \alpha_{\delta}(\beta) \right] / \nu_{\omega} = k, \quad \text{all } \beta \in B_{0} \cup B_{1}, \quad k = 1 \text{ or } 0,
\]

then also

\[
P_{\theta} - \lim_{a} \left[ -\log L(a) \right] / \nu_{\omega} = k.
\]

**Remarks.** It appears that in many cases, (5.8) is easier to verify than (5.4). Requirements (5.6) and (5.7) are actually equivalent to the following: there are critical values \( c_{\delta}(0) \) and \( c_{\delta}(1) \) so that

\[
\lim_{a} P_{\delta}(T(a) \geq c_{\delta}(1)) = 1 = \lim_{a} P_{\delta}(T(a) \leq c_{\delta}(0)) \quad \text{and} \quad \lim_{a} \left[ \log \alpha_{\delta}(0) \right] / \log \alpha_{\delta}(1) = 1.
\]

With a little care, it may be possible to check (5.10) directly, thus shortcutting the verifications necessary for Theorem 5.2. This theorem can be used to justify that suitable limits involving \( \log \alpha_{\delta} \) are, in fact, stochastic indices. We note also that Theorems 5.1 and 5.2 are not explicitly asymptotic: there is no (explicit) requirement that \( N_{\alpha} \to \infty \).

**Proof of Theorems 5.1 and 5.2.** Since \( \sup \{ P_{\omega}(T(a) \geq x) : \omega \in \Theta_{\omega} \} \) is nonincreasing in \( x \), we have that

\[
(5.11) \quad T(a) \geq c_{\delta} \to L(a) \leq \alpha_{\delta} \quad \text{and} \quad T(a) \leq c_{\delta} \to L(a) \geq \alpha_{\delta}.
\]

From (5.3) it follows that \( P_{\delta}(L(a) \leq \alpha_{\delta}) > \beta > 0 \) or that \( \beta < P_{\delta}([-\log L(a)] / \nu_{\omega} \geq [-\log \alpha_{\delta}] / \nu_{\omega}) \). Then, it follows from (5.4) that \( \lim \sup_{a} [-\log \alpha_{\delta}] / \nu_{\omega} \leq k \). Similarly, (5.3) and (5.11) entail \( \beta < P_{\delta}(T(a) < c_{\delta}) \leq P_{\delta}(L(a) \geq \alpha_{\delta}) \), from which it similarly follows that \( \lim \inf_{a} [-\log \alpha_{\delta}] / \nu_{\omega} \geq k \). Thus (5.5) holds.

Theorem 5.2 is proved similarly: from (5.7a) and (5.11) we obtain \( P_{\delta}([-\log L(a)] / \nu_{\omega} \geq [-\log \alpha_{\delta}(\beta)] / \nu_{\omega}) \geq \beta \), which, together with (5.8) entails \( \lim \inf_{a} P_{\delta}([-\log L(a)] / \nu_{\omega} > k - \varepsilon) \geq \beta \), all \( \varepsilon > 0 \), all \( \beta \in B_{1} \). Since \( \sup \{ \beta : \beta \in B_{1} \} = 1 \) (cf. (5.6)), it follows that \( P_{\theta} - \lim \inf_{a} [-\log L(a)] / \nu_{\omega} \geq k \). From (5.7b), one similarly concludes that \( P_{\delta} - \lim \sup_{a} [-\log L(a)] / \nu_{\omega} \leq k \), hence that (5.9) holds. \( \Box \)

We note that it is not always possible to obtain a stochastic index from significance levels. This is manifest in the nonsequential case if the stochastic index is random (not constant w.p. 1). Bahadur and Raghavachari (1972) discuss some nonsequential testing problems in which random stochastic indices occur.

In closing, we touch on the interpretation of a random stochastic index as a measure of efficiency. Suppose \( \{ T_{i}(a, n, a), N_{i, a} \} \), \( i = 1, 2 \) are two collections of test statistics and stopping times for the same data sequence. Let \( T_{i}(a), i = 1, 2 \) denote the corresponding stopped statistics. If, for \( i = 1, 2 \), \( L_{i}(a) \), the attained level of \( T_{i}(a) \), satisfies

\[
(5.12) \quad P - \lim_{a} [-\log L_{i}(a)] / N_{i, a} = r_{i},
\]
then the possibly random ratio \( r_1/r_2 \) does, in a sense, measure the asymptotic performance of \( T_1(a) \) to \( T_2(a) \): if

\[
P = \lim_a N_{2a}/N_{1a} = r_1/r_2, \tag{5.13}
\]

then the log-attained levels are equalized asymptotically. That is,

\[
P = \lim_a [\log L_1(a)]/[\log L_2(a)] = 1. \tag{5.14}
\]

That \( r_1/r_2 \) may not be constant then seems to mean that the asymptotic relative performance of \( T_1 \) to \( T_2 \) is different on different parts of the sample space. (If the convergence in (5.12) is w.p. 1, one could say that the relative performance is different for different data sequences.) In the sequential case, satisfying a condition like (5.13) with \( r_1/r_2 \) random seems to present no conceptual difficulties. However, an interpretation for the nonsequential case seems a bit elusive. If the convergence in (5.12) is w.p. 1, then choosing nonrandom sample sizes \( n_1 \) and \( n_2 \) so that \( n_2/n_1 \to \lambda \) means that the behaviors of \( T_1 \) and \( T_2 \) are matched (asymptotically) on the set of sample sequences for which \( r_1/r_2 = \lambda \). Of course, even when the convergence in (5.12) is only in probability, one has convergence w.p. 1 for a suitable subsequence.

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