Chapter 7. Statistical Intervals Based on a Single Sample

7.1 (Page 286)
   a. $z_{\alpha} = 2.81$ implies that $\frac{\alpha}{2} = 1 - \Phi(2.81) = .0025$, so $\alpha = .005$ and the confidence level is $100(1-\alpha)\% = 99.5\%$.
   b. $z_{\alpha} = 1.44$ for $\alpha = 2[1 - \Phi(1.44)] = .15$, and $100(1-\alpha)\% = 85\%$.
   c. 99.7% implies that $\alpha = .003$, $\frac{\alpha}{2} = .0015$, and $z_{.0015} = 2.96$. (Look for cumulative area .9985 in the main body of table A.3, the Z table.)
   d. 75% implies $\alpha = .25$, $\frac{\alpha}{2} = .125$, and $z_{.125} = 1.15$.

7.3 (Page 286)
   a. A 90% confidence interval will be narrower. The z critical value for a 90% confidence level is 1.645, smaller than the z of 1.96 for the 95% confidence level, thus producing a narrower interval.
   b. Not a correct statement. Once an interval has been created from a sample, the mean $\mu$ is either enclosed by it, or not. The 95% confidence is in the general procedure, for repeated sampling.
   c. Not a correct statement. The interval is an estimate for the population mean, not a boundary for population values.
   d. Not a correct statement. In theory, if the process were repeated an infinite number of times, 95% of the intervals would contain the population mean $\mu$.

7.7 (Page 287)
   If $L = 2z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ and we increase the sample size by a factor of 4, the new length is $L' = 2z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{4n}} = \left[ 2z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \right] \left( \frac{1}{2} \right) = \frac{L}{2}$. Thus halving the length requires $n$ to be increased fourfold. If $n' = 25n$, then $L' = \frac{L}{5}$, so the length is decreased by a factor of 5.
\[ n = 507, \; x = \text{# of successes} = 142 \Rightarrow \hat{p} = \frac{142}{507} = .28 \; ; \text{the 99\% two-sided interval is:} \]

\[
\frac{.28 + \left(\frac{2.58}{2(507)}\right) \pm 2.58}{1 + \left(\frac{2.58}{507}\right)^2} \sqrt{\frac{(28)(72)}{507} + \frac{(2.58)^2}{4(507)^2}} = \frac{.2866 \pm .0519}{1.0131} = (.2317, .3341)
\]

7.20 (New version Page 298)

\[ \hat{p} = \frac{24}{37} = .6486. \text{ The 99\% confidence interval for p is} \]

\[
\frac{.6486 + \left(\frac{2.58}{2(37)}\right) \pm 2.58}{1 + \left(\frac{2.58}{37}\right)^2} \sqrt{\frac{(6486)(.3514)}{37} + \frac{(2.58)^2}{4(37)^2}} = \frac{.7386 \pm .2216}{1.1799} = (.438, .814)
\]

7.23 (Page 294)

a. \[ \hat{p} = \frac{24}{37} = .6486. \text{ The 99\% confidence interval for p is} \]

\[
\frac{.6486 + \left(\frac{2.58}{2(37)}\right) \pm 2.58}{1 + \left(\frac{2.58}{37}\right)^2} \sqrt{\frac{(6486)(.3514)}{37} + \frac{(2.58)^2}{4(37)^2}} = \frac{.7386 \pm .2216}{1.1799} = (.438, .814)
\]

b. \[ n = \frac{2(2.58)^2(.25) - (2.58)^2(.01) \pm \sqrt{4(2.58)^4(.25)(.25 - .01) + .01(2.58)^4}}{.01}
\]

\[ = \frac{3.261636 \pm 3.3282}{.01} \approx 659 \]

7.25 (Page 295)

a. \[ n = \frac{2(1.96)^2(.25) - (1.96)^2(.01) \pm \sqrt{4(1.96)^4(.25)(.25 - .01) + .01(1.96)^4}}{.01}
\]

\[ \approx 381 \]

b. \[ n = \frac{2(1.96)^2\left(\frac{1}{3}, \frac{2}{3}\right) - (1.96)^2(.01) \pm \sqrt{4(1.96)^4\left(\frac{1}{3}, \frac{2}{3}\right)\left(\frac{1}{3}, \frac{2}{3} - .01\right) + .01(1.96)^4}}{.01}
\]

\[ \approx 338 \]

7.30 (Page 302)

a. \[ t_{.025,10} = 2.228 \quad \text{b.} \; t_{.025,15} = 2.131 \quad \text{c.} \; t_{.005,15} = 2.947 \]

\[ d. \; t_{.005,4} = 4.604 \quad \text{e.} \; t_{.01,24} = 2.492 \quad \text{f.} \; t_{.005,37} \approx 2.712 \]

7.35 (Page 303)

\[ n = 5, \; \bar{x} = 2887.6, \; s = 84.0; \; t_{.025,4} = 2.776 \]

a. \[ \text{A 95\% C.I. for the mean:} \; 2887.6 \pm (2.776)\left(\frac{84}{\sqrt{5}}\right) \Rightarrow (2783.3, 2991.9) \]
b. A 95% Prediction Interval: \( 2887.6 \pm 2.776(84) \sqrt{1 + \frac{1}{5}} \Rightarrow (2632.1, 3143.1) \). The P.I. is considerably larger than the C.I., about 2.5 times larger.

7.36 (Page 303)
\( n = 26, \bar{x} = 370.69, s = 24.36; t_{0.025} = 1.708 \)

a. A 95% upper confidence bound:
\[
370.69 + (1.708) \left( \frac{24.36}{\sqrt{26}} \right) = 370.69 + 8.16 = 378.85
\]

b. A 95% upper prediction bound:
\[
370.69 + 1.708(24.36) \sqrt{1 + \frac{1}{26}} = 370.69 + 42.45 = 413.14
\]

c. Following a similar argument as that on page 300 of the text, we need to find the variance of \( \bar{X} - \bar{X}_{new} \):
\[
V(\bar{X} - \bar{X}_{new}) = V(\bar{X}) + V(\bar{X}_{new}) = V(\bar{X}) + V\left( \frac{1}{2} X_{27} + X_{28} \right)
\]
\[
= V(\bar{X}) + V\left( \frac{1}{2} X_{27} \right) + V\left( \frac{1}{2} X_{28} \right) = V(\bar{X}) + \frac{1}{4} V(X_{27}) + \frac{1}{4} V(X_{28})
\]
\[
= \frac{\sigma^2}{n} + \frac{1}{4} \sigma^2 + \frac{1}{4} \sigma^2 = \sigma^2 \left( \frac{1}{2} + \frac{1}{n} \right).
\]
We eventually arrive at \( T = \frac{\bar{X} - \bar{X}_{new}}{s \sqrt{\frac{1}{2} + \frac{1}{n}}} - t \) distribution with \( n - 1 \) d.f., so the new prediction interval is \( \bar{x} \pm t_{a/2,n-1} \cdot s \sqrt{\frac{1}{2} + \frac{1}{n}} \).
For this situation, we have
\[
370.69 \pm 2(24.36) \sqrt{1 + \frac{1}{26}} = 370.69 \pm 36.68 = (333.87, 407.51)
\]

Additional Three Problems: A, B & C

A. \( \hat{p} = 0.43 \). The 90% confidence interval for \( p \) is
\[
0.43 + \frac{(1.65)^2}{2(611)} \pm 1.65 \sqrt{\frac{0.43(0.57)}{611} + \frac{(1.65)^2}{4(611)^2}} = 0.4322 \pm 0.00331 = (0.397, 0.463)
\]

\[ a. \]
\[ b. \] The population is all the people who heard the speech.
\[ c. \] The margin of error of this survey with 95% confidence is wider than 3%.

\[
\frac{4(1.96)^2(0.43)(0.57)}{(0.06)^2} = 1047
\]

B. \( n = 20, \bar{x} = 87.40, s = .52; t_{0.025,19} = 2.093 \). By formula (7.15) on book page 298,
\[ a. \] 95% confidence interval: \( (87.15, 87.64) \).
\[ b. \] By Figure 1’s Normal Quantile Plot, the assumption of normality seems plausible.

C. \[ a. \] \( Y \) has the distribution of binomial with \( n=100, p=0.95 \). So:
\[
E(Y) = 95, \sigma(Y) = \sqrt{100(0.95)(0.05)} = 2.18
\]
b. \( P(Y \geq 90) \approx P(Z \geq \frac{90 - 95}{2.18}) = 0.989 \)

c. \( P(Y \geq 95) \approx P(Z \geq \frac{95 - 95}{2.18}) = 0.5 \)

d. \( P(Y \geq 99) \approx P(Z \geq \frac{99 - 95}{2.18}) = 0.033 \)