1. Let $Y$ have a binomial distribution with index $n$ and parameter $\theta$. In class the “best” estimator of $\theta^2$ was found, and its variance was stated. Derive the variance given in class, using the facts that if $Y$ has this binomial distribution,

$$E(Y^4) = n(n-1)(n-2)(n-3) \theta^4 + 6n(n-1)(n-2) \theta^3 + 7n(n-1) \theta^2 + n \theta,$$

$$E(Y^3) = n(n-1)(n-2) \theta^3 + 3n(n-1) \theta^2 + n \theta,$$

$$E(Y^2) = n(n-1) \theta^2 + n \theta, \quad E(Y) = n \theta.$$

2. (Continuation from Question 1). Carry out the parallel procedures for the “best” estimator of $\theta(1-\theta)$. That is, find this “best” estimator, (Hint: use “Approach 1”), find the variance of this estimator, find the Cramér-Rao lower bound for the variance of any unbiased estimator of $\theta(1-\theta)$, and then compare this bound with the variance of the “best” estimator.

3. Suppose that $Y_i$ has the Poisson distribution with parameter $\theta$ and put $Y = Y_1 + Y_2 + \ldots + Y_n$, where $n > 4$. Then (of course) $Y$ has the Poisson distribution with parameter $n \theta$. It was shown in class that the MVU estimator of $\theta^2$ is $(Y/n)^2 - Y/n^2$. By using the facts that

$$E(Y^4) = (n\theta)^4 + 6(n\theta)^3 + 7(n\theta)^2 + n\theta, \quad E(Y^3) = (n\theta)^3 + 3(n\theta)^2 + n\theta,$$

$$E(Y^2) = (n\theta)^2 + (n\theta), \quad E(Y) = n\theta,$$

find the variance of this estimator. We know from theory that this variance must exceed the Cramér-Rao bound for the variance of any unbiased estimator of $\theta^2$. Find this Cramér-Rao bound and hence find the value of this excess.

4. (Continuation from Question 3). Suppose that in the situation of Question 3 we want to find the MVU estimator of $e^{3\theta}$. Use “Approach 2” and the same sort of argument that was used in class to find the “best” estimator of $e^\theta$ and $e^{2\theta}$ to find this “best” estimator.

5. For extra credit. (Continuation from Question 4). Find the variance of the “best” estimator found in question 4 and compare it to the relevant Cramér-Rao bound.
6. Suppose that $Y_1, Y_2, \ldots, Y_n$ are NID(0, $\sigma^2$) and we wish to find the MVU estimator of $\sigma^4$. Define $S^2$ by $S^2 = \sum_{j=1}^{n} Y_j^2$. Then we know that $S^2$ is a sufficient statistic for $\sigma^2$. Use the fact that for any positive integer $j$,

$$E(S^{2j}) = \sigma^{2j} n(n+2)(n+2(j-1))$$

to (i) find the "best" estimator of $\sigma^4$ and (ii) to find the variance of this estimator. (Hint: Use "Approach 1". That is, start with $S^4$ as a first try, and then look for an estimator of the form $k S^4$, for some constant $k$, which is an unbiased estimator of $\sigma^4$.) Compare this variance with the Cramér-Rao bound for the variance of any unbiased estimator of $\sigma^4$.

7. (Harder, and for extra credit). Suppose that each $Y_i$ has the uniform distribution $f_Y(y) = \theta^{-1}$, $0 < y < \theta$. In Homework 3 question 3 you showed that both $(n+1)Y_{(1)}$ and $((n+1)/n)Y_{(n)}$ are unbiased estimators of $\theta$. You found the variances of both estimators, and from this you found that $((n+1)/n)Y_{(n)}$ is "better" than $(n+1)Y_{(1)}$. Now we know that $Y_{(n)}$ is a sufficient statistic for $\theta$, and the Rao-Blackwell theorem then shows that $((n+1)/n)Y_{(n)}$ is the "best" estimator of $\theta$. However, since $(n+1)Y_{(1)}$ is also an unbiased estimator of $\theta$, the Rao-Blackwell theorem shows that the expected value of $(n+1)Y_{(1)}$ GIVEN $Y_{(n)}$ must also be the "best" estimator $\theta$, implying that it must be identical to $((n+1)/n)Y_{(n)}$. Prove that this is so, using the conditional density function of $Y_{(1)}$ given $Y_{(n)}$. (HINT: This conditional density function is the joint density function of $Y_{(1)}$ and $Y_{(n)}$ divided by the density function of $Y_{(n)}$, both of which you either already know or can easily work out from order statistics theory.)