STATISTICS 512

Mid-term exam, 18 March 2010

Notes: (a) This is a long exam and you are not expected to finish it. Anyone finishing it (including the “extra credit” questions) will score about 130 points out of 100.

(b) Some questions are harder than others and thus carry more points. It is part of the examining process for you to assess which are the harder questions.

1. If $Y_1$, $Y_2$, ..., $Y_n$ are iid random variables coming from a uniform continuous distribution over $[-1,1]$, find the density function of the smallest order statistic ($Y_{(1)}$), using a formula given in class. From your answer, find by an integration process the probability that $Y_{(1)}$ takes a positive value. How might your answer have been found almost immediately?

2. If $Y_1$, $Y_2$, ..., $Y_n$ are iid random variables coming from a uniform distribution over the interval $[0,1]$, find the joint density function of the smallest ($Y_{(1)}$) and the largest ($Y_{(n)}$) order statistics. From your answer, find the density function of the random variable $R$, defined by $R = Y_{(n)} - Y_{(1)}$. [NOTE: This year (2011) we have not yet done the theory that would enable you to answer the last part of this question.]

3. What do you understand by the statement that “$Y$ is an unbiased estimator of $g(\theta)$”, (where $g(\theta)$ is some specified function of a parameter $\theta$)? If $Y_1$, $Y_2$, ..., $Y_n$ are iid random variables coming from a Poisson distribution with parameter $\theta$, and $\bar{Y} = (Y_1 + Y_2 + ... + Y_n)/n$, find some function of $\bar{Y}$ which is an unbiased estimator of $\theta^2$.

4. Suppose that $Y_1$, $Y_2$, ..., $Y_n$ are NID(0, $\sigma^2$) random variables. Use “equation 24” (the equation which indicates when there exists some function of a parameter having an unbiased estimator whose variance achieves the Cramér-Rao bound) and its consequences to find the function of $\sigma^2$ which, up to an arbitrary multiplicative constant, admits an unbiased estimator whose variance achieves the Cramér-Rao bound. Also find the function of $Y_1$, $Y_2$, ..., $Y_n$ which is an unbiased estimator of $\sigma^2$ whose variance achieves this Cramér-Rao bound.

5. What are the deficiencies of the Cramér-Rao bound approach to finding minimum variance unbiased (MVU) estimators of some function of a parameter $\theta$? How is one of these deficiencies overcome by using the Rao-Blackwell approach?

6. What do you understand by the concept of a “non-trivial minimal sufficient statistic” for a parameter?

7. If $Y_1$, $Y_2$, ..., $Y_n$ are iid continuous random variables, each having density function

$$f(y) = \left[2y/\theta\right] \, e^{-y^2/\theta}, \quad 0 < y,$$
\[ f(y) = 0 \quad \text{elsewhere}, \]

use the (i) factorization, (ii) “Smith-Jones” and (iii) exponential family approaches to find a non-trivial minimal sufficient statistic for \( \theta \).

*For extra credit.* Use “Approach 1” (that is, starting with some function of the sufficient statistic for \( \theta \)) deriving from the Rao-Blackwell theorem to find the minimum variance unbiased estimator of \( \theta \).

8. If \( Y_1, Y_2, ..., Y_n \) are iid random variables coming from a Poisson distribution with parameter \( \theta \), outline without any mathematics, and without doing any calculations at all, how you could use “Approach 2” (that is, starting with any unbiased estimator) deriving from the Rao-Blackwell theorem to find the minimum variance unbiased estimator of \( \theta^3 e^{-\theta} \).

9. Suppose that \( Y_1, Y_2, ..., Y_n \) are iid continuous random variables, each having probability distribution

\[
f(y) = \frac{2y}{\theta^3}, \quad 0 < y < \theta
\]

\[
f(y) = 0 \quad \text{elsewhere.}
\]

Find the minimal non-trivial sufficient statistic for \( \theta \). Use “Approach 1” (that is, starting with some function of the sufficient statistic for \( \theta \)) deriving from the Rao-Blackwell theorem to find the non-trivial minimum variance unbiased estimator of \( \theta \).