STAT 430/510 Probability
Lecture 16, 17: Compute by Conditioning

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Lecture 16 - 17

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A is an arbitrary event

If $Y$ is a discrete random variable,  
$$P(A) = \sum_y P(A|Y = y)P(Y = y)$$
It is just the rule of total probability.
Computing Probability by Conditioning

- $A$ is an arbitrary event
- If $Y$ is a discrete random variable,
  \[ P(A) = \sum_y P(A|Y = y)P(Y = y) \]
  It is just the rule of total probability.
- If $Y$ is a continuous random variable,
  \[ P(A) = \int_{-\infty}^{\infty} P(A|Y = y)f_Y(y)\,dy \]
Example

Let $U$ be a uniform random variable on $(0,1)$, and suppose that the conditional distribution of $X$, given that $U = p$, is binomial with parameters $(n, p)$. Find the probability that $X = 0$. 
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Let $U$ be a uniform random variable on $(0,1)$, and suppose that the conditional distribution of $X$, given that $U = p$, is binomial with parameters $(n, p)$. Find the probability that $X = 0$.

\[
P(X = 0) = \int_0^1 P(X = 0 | U = p) f_U(p) dp
\]

\[
= \int_0^1 P(X = 0 | U = p) dp
\]

\[
= \int_0^1 (1 - p)^n dp
\]
The conditional expectation of $X$ given $Y = y$ is defined as

$$E(X|Y) = \sum_x xP(X = x|Y) \ (Discrete)$$

or

$$E(X|Y) = \int_x xf_{X|Y}(x|Y)dx \ (Continuous).$$
If $Y$ is a discrete random variable,
$$E[X] = \sum_y E[X \mid Y = y] P(Y = y)$$

If $Y$ is a continuous random variable,
$$E[X] = \int_y E[X \mid Y = y] f_Y(y) dy$$
Computing Expectations by Conditioning

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$$E[X] = E[E[X \mid Y]]$$
Example

There are some bulbs in a box. 30% of them are style A bulbs, which can last 10 hours with standard deviation 1 hour; the other are style B bulbs, which can last 15 hours with standard deviation 2 hour. If you choose on bulb randomly, what is the expectation of its lifetime?
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\[ E(X) = E(E(X|\text{style})) = 10 \times 30\% + 15 \times 70\% = 13.5. \]
Case 1: Expectation of a Sum of a Random Number of R.V.’s

Suppose that the number of people entering a department store on a given day is a random variable with mean 50. Suppose further that the amounts of money spent by these customers are independent random variables having a common mean of 8 dollars. Finally, suppose also that the amount of money spent by a customer is also independent of the total number of customers who enter the store. What is the expected amount of money spent in the store on a given day?
Example: Solution

- Let $N$ denote the number of customers that enter the store and $X_i$ be the amount spent by the $i$th customer.
- $E[\sum_{i=1}^{N} X_i] = E[E[\sum_{i=1}^{N} X_i | N]]$
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- $E[\sum_{i=1}^{N} X_i] = E[E[\sum_{i=1}^{N} X_i | N]]$
  
  Since $E[\sum_{i=1}^{N} X_i | N = n] = E[\sum_{i=1}^{n} X_i] = nEX_1$, we have $E[\sum_{i=1}^{N} X_i | N] = NEX_1$. 
Example: Solution

- Let $N$ denote the number of customers that enter the store and $X_i$ be the amount spent by the $i$th customer.

$$E[\sum_{i=1}^N X_i] = E[E[\sum_{i=1}^N X_i | N]]$$

Since $E[\sum_{i=1}^N X_i | N = n] = E[\sum_{i=1}^n X_i] = nE[X_1]$, we have $E[\sum_{i=1}^N X_i | N] = NEX_1$.

Thus $E[\sum_{i=1}^N X_i] = E[NEX_1] = EN \times E[X_1] = 50 \times 8 = 400$
Case 2: Method of Indicator: Recall the shooting duck example

10 hunters are waiting for ducks to fly by. When a flock of ducks flies overland, the hunters fire at the same time, but each chooses his target at random, independently of the others. Each hunter independently hit his target with probability $p$. We’ve already computed the expected number of ducks got hit when a flock of size $k$ flies overhead to be:

Given $k$ ducks, $E[X] = k[1 - (1 - \frac{p}{k})^{10}]$. 

Now suppose the number of ducks in this flock follows a Geometric distribution with mean 12. What is the expected number of duck got hit?

$E[X] = E(E[X|\text{number of ducks}]) = \sum_{k=1}^{\infty} k \left[1 - (1 - \frac{p}{k})^{10}\right] \cdot P(k \text{ ducks}) = \left(1 - \frac{1}{12}\right)^k - 1 \cdot \frac{1}{12}$. 


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- Now suppose the number of ducks in this flock follows a Geometric distribution with mean 12. What is the expected number of duck got hit? 
  $E[X] = E(E(X|number \ of \ ducks)) = \sum_{k=1}^{\infty} k[1 - (1 - \frac{p}{k})^{10}] \times P(k \ ducks)$, 
  where $P(k \ ducks) = (1 - \frac{1}{12})^{k-1} \frac{1}{12}$. 

Case 3: Another Example which is a little confusing

A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third leads to a tunnel that will return him to the mine after 7 hours. If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety?
Example: Solution

- Let $X$ denote the amount of time until the miner reaches safety, and let $Y$ denote the door he initially chooses.
Example: Solution

Let $X$ denote the amount of time until the miner reaches safety, and let $Y$ denote the door he initially chooses.


$$E[X|Y = 1] = 3, \; E[X|Y = 2] = 5 + E[X], \; E[X|Y = 3] = 7 + E[X]$$
Example: Solution

- Let $X$ denote the amount of time until the miner reaches safety, and let $Y$ denote the door he initially chooses.


- $P(Y = 1) = P(Y = 2) = P(Y = 3) = 1/3$
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- $E[X|Y=1] = 3, \ E[X|Y=2] = 5 + E[X], \ E[X|Y=3] = 7 + E[X]$

- $P(Y=1) = P(Y=2) = P(Y=3) = 1/3$

- $E[X] = 15$
The conditional variance of $X$ given $Y = y$ is defined as

$$Var(X|Y) = E[(X - E[X|Y])^2 | Y]$$

But most of the time we just use the intuitive. $Var(X|Y)$ is just the variance of $X$, but under the condition that $Y$ is given.
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The conditional variance formula:

$$Var(X) = E[Var(X|Y)] + Var(E[X|Y])$$
Case 1:

There are some bulbs in a box. 30% of them are style A bulbs, which can last 10 hours with standard deviation 1 hour; the other are style B bulbs, which can last 15 hours with standard deviation 2 hour. If you choose on bulb randomly, what is the standard deviation of its lifetime?

\[ \text{Var}(X) = E(\text{Var}(X|\text{style})) + \text{Var}(E(X|\text{style})). \]
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\[ \text{Var}(X) = E(\text{Var}(X|\text{style})) + \text{Var}(E(X|\text{style})). \]

\[ \text{Var}(X|\text{style A}) = 1, \text{Var}(X|\text{style B}) = 4, \text{thus} \]
\[ E(\text{Var}(X|\text{style})) = 0.3 \times 1 + 0.7 \times 4 = 3.1. \]
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  \[E(\text{Var}(X|\text{style})) = 0.3 \times 1 + 0.7 \times 4 = 3.1.\]

- \(E(X|\text{style A}) = 10, E(X|\text{style B}) = 15\), thus
  \[\text{Var}(E(X|\text{style})) = 0.3 \times [10 - (0.3 \times 10 + 0.7 \times 15)]^2 + 0.7 \times [15 - (0.3 \times 10 + 0.7 \times 15)]^2 = 5.25.\]

\[\text{SD}(X) = \sqrt{\text{Var}(X)} = \sqrt{3.1 + 5.25} = 2.89.\]
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Case 2: Variance of Sum of a Random Number of R.V.’s

Suppose there are N people in household, and each person’s income independently and identically follows a Normal distribution with mean 30,000 dollars and standard deviation 5,000 dollars. N follows a Geometric distribution with mean 3. What is the expected value of the total income, and the variance?
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Let \( X \) represent the total income. Then
\[
E(X) = E(E(X|N)) = E(30,000N) = 90,000 \text{ dollars.}
\]
Case 2: Variance of Sum of a Random Number of R.V.’s

Suppose there are $N$ people in household, and each person’s income independently and identically follows a Normal distribution with mean 30,000 dollars and standard deviation 5,000 dollars. $N$ follows a Geometric distribution with mean 3. What is the expected value of the total income, and the variance?

Let $X$ represent the total income. Then

$$E(X) = E(E(X|N)) = E(30,000N) = 90,000 \text{ dollars}.$$ 

$$Var(X) = E(Var(X|N)) + Var(E(X|N)) = E(5,000^2N) + Var(30,000N) = 5,000^2 \times 3 + 30,000^2 \times \frac{1-1/3}{(1/3)^2}.$$
Let \( X_1, X_2, \cdots \) be a sequence of independent and identically distributed random variables, and let \( N \) be a nonnegative integer-valued random variable that is independent of the sequence \( X_i, i \geq 1 \). Then,

\[
\text{Var} \left( \sum_{i=1}^{N} X_i \right) = E[N] \text{Var}(X) + (E[X])^2 \text{Var}(N)
\]
A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third leads to a tunnel that will return him to the mine after 7 hours. If we assume that the miner is at all times equally likely to choose any one of the doors, what is the variance of the length of time until he reaches safety?
Example

\[ \text{Var}(X) = E(\text{Var}(X|Y)) + \text{Var}(E(X|Y)) \]
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- \( \text{Var}(X) = E(\text{Var}(X|Y)) + \text{Var}(E(X|Y)) \)
- \( \text{Var}(X|Y = 1) = 0; \text{Var}(X|Y = 2) = \text{Var}(X); \text{Var}(X|Y = 3) = \text{Var}(X). \)

So

\[
E(\text{Var}(X|Y)) = 0 \cdot \frac{1}{3} + \text{Var}(X) \cdot \frac{1}{3} + \text{Var}(X) \cdot \frac{1}{3} = \frac{2}{3} \text{Var}(X).
\]
Example

- $\text{Var}(X) = E(\text{Var}(X|Y)) + \text{Var}(E(X|Y))$

- $\text{Var}(X|Y = 1) = 0$; $\text{Var}(X|Y = 2) = \text{Var}(X)$; $\text{Var}(X|Y = 3) = \text{Var}(X)$.

  So

  $E(\text{Var}(X|Y)) = 0 \times \frac{1}{3} + \text{Var}(X) \times \frac{1}{3} + \text{Var}(X) \times \frac{1}{3} = \frac{2}{3} \text{Var}(X)$.


  We have computed that $EX = 15$ in a previous example. So $E[X|Y = 1] = 3$, $E[X|Y = 2] = 20$, $E[X|Y = 3] = 22$.

  Thus $\text{Var}(E(X|Y)) = 218/3$. 


Example

- \( \text{Var}(X) = E(\text{Var}(X|Y)) + \text{Var}(E(X|Y)) \)
- \( \text{Var}(X|Y = 1) = 0; \text{Var}(X|Y = 2) = \text{Var}(X); \text{Var}(X|Y = 3) = \text{Var}(X). \)
  So
  \[ E(\text{Var}(X|Y)) = 0 \times \frac{1}{3} + \text{Var}(X) \times \frac{1}{3} + \text{Var}(X) \times \frac{1}{3} = \frac{2}{3} \text{Var}(X). \]
- \( E[X|Y = 1] = 3, \ E[X|Y = 2] = 5 + E[X], \ E[X|Y = 3] = 7 + E[X]. \)
  We have computed that \( EX = 15 \) in a previous example.
  So \( E[X|Y = 1] = 3, \ E[X|Y = 2] = 20, \ E[X|Y = 3] = 22. \)
  Thus \( \text{Var}(E(X|Y)) = 218/3. \)
- Thus \( \text{Var}(X) = \frac{2}{3} \text{Var}(X) + 218/3 \Rightarrow \text{Var}(X) = 218. \)
All the above stuff can also be computed using:

- \( \text{Var}(X) = E(X^2) - (EX)^2. \)
- \( E(X^2) \) can be computed by conditioning.
- The bulb example in Case 1.