A sample spaces $S$ of a random experiment is the set of all possible outcomes.

An event $E$ is any subset of the sample space $S$.

Our objective is to determine $P(E)$, the probability that event $E$ will occur.

Events and sample spaces can be quite complex, and the each outcome may not necessarily be equally likely to happen.
The **union** of two events $A$ and $B$, $A \cup B$, is the event consisting of all outcomes that are either in $A$ or in $B$ or in both events.

The **complement** of an event $A$, $A^c$, is the set of all outcomes in $S$ that are not in $A$.

The **intersection** of two events $A$ and $B$, $A \cap B$ or $AB$, is the event consisting of all outcomes that are in both events.

When two events $A$ and $B$ have no outcomes in common, $AB = \emptyset$, they are said to be **mutually exclusive**, or **disjoint**, events.
Example

- The experiment: toss a coin 10 times and the number of heads is observed.
- Let $A = \{0, 2, 4, 6, 8, 10\}$, $B = \{1, 3, 5, 7, 9\}$, $C = \{6, 7, 8, 9, 10\}$.
- $A \cup B = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = S$
- $AB = \emptyset$. So A and B are mutually exclusive.
- $C^c = \{0, 1, 2, 3, 4, 5\}$, $AC^c = \{0, 2, 4\}$
Rules

- **Commutative Laws:** \( A \cup B = B \cup A \), \( AB = BA \)
- **Associative Laws:** \((A \cup B) \cup C = A \cup (B \cup C)\), \((AB)C = A(BC)\)
- **Distributive Laws:** \((A \cup B) C = AC \cup BC\), \((AB) \cup C = (A \cup C)(B \cup C)\)
- **DeMorgan’s Laws:** \((\bigcup_{i=1}^{n} A_i)^c = \bigcap_{i=1}^{n} A_i^c\), \((\bigcap_{i=1}^{n} A_i)^c = \bigcup_{i=1}^{n} A_i^c\)
- These laws can be shown by **Venn diagram**.
Probability Distribution

- Probability of an event can be interpreted as the limiting relative frequency of the event.
- Probabilities satisfy the following axioms.
Axioms of Probability

- **Axiom 1**: $0 \leq P(E) \leq 1$
- **Axiom 2**: $P(S) = 1$
- **Axiom 3**: For any sequence of mutually exclusive events $E_1, E_2, \ldots$, (that is, events for which $E_i E_j = \emptyset$ when $i \neq j$),

  \[
  P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)
  \]

These axioms of probability require no proof.
Example

- A fair die is rolled.
- \( S = \{1, 2, 3, 4, 5, 6\} \).
  - Using **Axiom 2**: \( P(\{1, 2, 3, 4, 5, 6\}) = 1 \)
  - \( P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\}) = \frac{1}{6} \).
  - Using **Axiom 3**: \( P(\{2, 4, 6\}) = \frac{1}{2} \)
Properties of Probability

- $P(E^c) = 1 - P(E)$
- If $E \subset F$, then $P(E) \leq P(F)$
- $P(E \cup F) = P(E) + P(F) - P(EF)$
A fair die is rolled.

$S = \{1, 2, 3, 4, 5, 6\}; \ E = \{2, 4, 6\}.$

We’ve known $P(E) = \frac{1}{2}$. So

$P(E^c) = P(\{1, 3, 5\}) = 1 - \frac{1}{2} = 0.5.$
Example

- J is taking two books along on her holiday vacation. With probability 0.5, she will like the first book; with probability 0.4, she will like the second book; and with probability 0.3, she will like both books. What is the probability that she likes neither book?

- Let $B_i$ denote the event that J likes book $i$, $i = 1, 2$.

$$P(B_1^c B_2^c) = P((B_1 \cup B_2)^c) = 1 - P(B_1 \cup B_2) = 1 - (P(B_1) + P(B_2) - P(B_1 B_2)) = 0.4$$
Example

- A town has 3 newspapers: A, B, and C.
- Proportions of people who read these papers:
  - A: 10%; B: 30%; C: 50%
  - AB: 8%; AC: 2%; BC: 4%
  - ABC: 1%.
- What is the portion of people who read only 1 newspaper?
- Portion of people who do not read any newspaper?
Inclusion-exclusion Identity

\[ P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \cdots + (-1)^{r+1} \sum_{i_1 < i_2 < \cdots < i_r} P(E_{i_1} E_{i_2} \cdots E_{i_r}) + \cdots + (-1)^{n+1} P(E_{i_1} E_{i_2} \cdots E_{i_n}) \]
n people put their ID’s together, and then they choose one ID randomly. What is the probability that none of them get the right ID?
Inclusion-exclusion Identity

Use Inclusion-exclusion Identity!

- Set \( E_i \) to be the event ’the ith person gets the right ID’.
- Then the event \( E = \{\text{none of them gets the right ID}\} = \{\text{some of them get the right ID}\}^c = (\bigcup E_i)^c \)
- So \( P(E) = 1 - P(\bigcup E_i) \).
Inclusion-exclusion Identity

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- Set $E_i$ to be the event ‘the $i$th person gets the right ID’.
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P(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \cdots + (-1)^{r+1} \sum_{i_1 < i_2 < \cdots < i_r} P(E_{i_1} E_{i_2} \cdots E_{i_r}) + \cdots + (-1)^{n+1} P(E_{i_1} E_{i_2} \cdots E_{i_n})
\]
Use Inclusion-exclusion Identity!

\[ P(E_{i_1} E_{i_2} \cdots E_{i_r}) = \frac{(n-r)!}{n!} \]
Use Inclusion-exclusion Identity!

- \( P(E_{i_1} E_{i_2} \cdots E_{i_r}) = \frac{(n-r)!}{n!} \)
- The terms like \((E_{i_1} E_{i_2} \cdots E_{i_r})\) have \(\binom{n}{r}\) choices.
Use Inclusion-exclusion Identity!

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- So \(\sum_{i_1 < i_2 < \cdots < i_r} P(E_{i_1} E_{i_2} \cdots E_{i_r}) = \frac{(n-r)!}{n!} \binom{n}{r} = \frac{1}{r!}.\)
Inclusion-exclusion Identity

Use Inclusion-exclusion Identity!

- $P(E_{i_1} E_{i_2} \cdots E_{i_r}) = \frac{(n-r)!}{n!}$

- The terms like $(E_{i_1} E_{i_2} \cdots E_{i_r})$ have $\binom{n}{r}$ choices.

- So $\sum_{i_1 < i_2 < \cdots < i_r} P(E_{i_1} E_{i_2} \cdots E_{i_r}) = \frac{(n-r)!}{n!} \binom{n}{r} = \frac{1}{r!}$.

- So $P(\cup E_i) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^n}{n!}$. 
Inclusion-exclusion Identity

Use Inclusion-exclusion Identity!

- $P(E_{i_1} E_{i_2} \cdots E_{i_r}) = \frac{(n-r)!}{n!}$

- The terms like $(E_{i_1} E_{i_2} \cdots E_{i_r})$ have $\binom{n}{r}$ choices.

- So $\sum_{i_1<i_2<\cdots<i_r} P(E_{i_1} E_{i_2} \cdots E_{i_r}) = \frac{(n-r)!}{n!} \binom{n}{r} = \frac{1}{r!}$.

- So $P(\bigcup E_i) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots + \frac{(-1)^n}{n!}$.

- Thus $P(E) = 1 - P(\bigcup E_i) = \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n+1}}{n!}$.