Fractional insurance: strategies to deal with huge potential losses

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Abstract

This paper describes and compares different optimization criteria for choosing fractional insurance coverage for risky ventures that may incur large losses and have high insurance rates due to substantial chances of failure. Four criteria are included: minimax loss, minimum expected loss, minimum expected loss with a limit on maximum loss, and maximum expected utility. The discussion is illustrated by satellite missions that can incur losses in excess of $100 million.

1. Introduction

Insurance decisions reflect attitudes toward risks and losses. Full insurance against the worst possible contingency indicates extreme conservatism, while hedges and deductibles that accept partial risks at lower policy costs reveal more moderate attitudes. This paper is concerned about alternative attitudes toward insurance that are modeled by specific decision rules or criteria. Its purpose is to describe and compare alternative criteria for insuring risky ventures that have substantial probabilities for incurring huge losses. A realistic example is insurance to hedge against losses in the neighborhood of $100 million to $200 million for a satellite’s launching and operation. We use a formulation which allows insurance to be taken on any fraction of total asset value and consider optimal insurance fractions under four decision criteria that range from the conservative minimax loss rule to maximization of expected utility.

The reason for considering several criteria is that the financial situations of agencies or companies and their perspectives on the severity of large losses can differ greatly from situation to situation. For example, aside from embarrassing publicity, a government might easily sustain a loss of $100 million that would be catastrophic for a small company. Our comparative analysis of alternative criteria is designed to help decision makers assess which criterion is most suitable for a particular situation. As we proceed through the criteria, we identify their different information requirements, their philosophies for dealing with potential losses, and the mathematical aspects of their applications. Although the paper’s methodology is widely applicable, we focus on a satellite mission to illustrate the discussion.

The formulation we adopt is based on five parameters for a single venture:
\[A = \text{asset value},\]
\[r = \text{insurance rate},\]
\[T = \text{tax rate},\]
\[L = \text{fractional loss of asset value},\]
\[F = \text{insured fraction of asset value}.\]

Parameters \(A\), \(r\), and \(T\) are treated as givens. Realistic values for a corporate satellite mission are \(A = \$200\) million, \(r = 0.20\), and \(T = 0.40\). The relatively large value of \(r\) is a consequence of the substantial chance of incurring total or partial loss: the mission might be aborted in the launch stage, or a successful launch might be followed by a malfunction that truncates the mission’s potential value. The tax rate \(T\) of 40 per cent may be realistic for a write-off of a corporate loss, but would be substantially smaller if not zero for a government-run mission.

Parameter \(L\) is a random variable that ranges from 0 for a completely successful mission to 1 for a total loss. Intermediate values describe a partly successful mission. \(F\) is the decision variable: \(F = 0\) denotes no insurance, \(F = 1\) denotes full insurance, and \(0 < F < 1\) signifies fractional insurance between the two extremes.

As noted below, our criteria presume different information about the probability distribution \(P\) on \(L\). The minimax loss rule requires no information about \(P(L)\), whereas full assessment of \(P\), based on engineering judgments and actuarial data, may be needed for maximization of expected utility.

We denote by \(g(F, L)\) the net financial loss of a risky venture when \(F\) is chosen as the fractional insurance coverage and \(L\) occurs as the fractional loss. Briefly
\[g(F, L) = \text{insurance premium} - \text{insurance loss recovery} + \text{loss of uninsured asset} - \text{tax savings}.\]

If \(F\) is the fraction of \(A\) to be insured, the unadjusted insurance premium would be \(rFA\). However, full recovery of premium in the event of a loss would elevate the premium to \(rFA + rF(rFA) + rF(r^2F^2A) + \cdots = rF[A/(1 - rF)]\). We assume that full recovery is desirable and therefore use the premium-adjusted gross asset value \(A/(1 - rF)\) rather than \(A\) to price out the components of \(g\). Let \(A^* = A/(1 - rF)\). Then
\[
\begin{align*}
\text{insurance premium} &= rFA^*, \\
\text{insurance loss recovery} &= L(rFA^*), \\
\text{loss of uninsured asset} &= L(1 - F)A^*, \\
\text{tax savings} &= T[L(1 - F)A^*].
\end{align*}
\]

Aggregation yields
\[g(F, L) = a(F) + Lb(F),\]
where
\[
\begin{align*}
a(F) &= ArF/(1 - rF), \\
b(F) &= A[(1 - F)(1 - T) - rF]/(1 - rF).
\end{align*}
\]

In this aggregate form, \(a(F)\) is the net insurance premium and \(Lb(F)\), which can be negative, indicates further expenditures \((b > 0)\) or gain \((b < 0)\) that can accrue from
the choice of $F$ coupled with fractional loss $L$. The fact that $g$ is linear in $L$ will play an important role in our analysis.

The possibility that $b$ can be negative is due to our assumption of unrestricted full recovery of premium in the event of a loss. This is atypical for many insurance markets which, for incentive reasons, often avoid insurance contracts where the insured can gain from a larger loss. When $b < 0$ is prevented by changes in the formulation, simplifications occur in the analysis of the criteria examined below. For example, if $b(F)$ is always positive, then the opportunity set for the expected utility criterion depends only on the mean and standard deviation of $g$, as determined by the loss probabilities. Differences among decision makers can then be captured by differences in their induced preferences as a function of mean and standard deviation. We refer readers to Meyer (1987) for further details of this approach.

Our four criteria for determining $F$ are as follows:

1. minimize maximum loss (Fishburn, 1964; Savage, 1951; Wald, 1950),
2. minimize expected loss,
3. minimize expected loss subject to a limit on maximum loss (Banerjee, 1964; Roy, 1952),

Written as functions of $F$, the maximum loss is

$$\max_L g(F, L) = \begin{cases} a(F) + b(F) & \text{if } b(F) \geq 0 \text{ [set } L = 1 ] \\ a(F) & \text{if } b(F) < 0 \text{ [set } L = 0 ] \end{cases}$$

the expected loss is

$$E(g, F) = \int_0^1 g(F, L)dP(L) = a(F) + b(F)E(L),$$

and the expected utility is

$$E(u, F) = \int_0^1 u(g(F, L))dP(L),$$

where $P$ is a probability measure for $L$ on $[0, 1]$, $E(L) = \int LdP(L)$ is the expected fractional loss of the venture, and $u$ is a utility function for net financial loss.

The information needed to apply the criteria increases as we go from criterion 1 to criterion 4. Criterion 1 requires only $g$, $a$, and $b$ to determine the minimum over $F$ of (1) and the corresponding minimax loss amount. Criterion 2 also requires an estimate of $E(L)$, perhaps without detailed consideration of $P$, to determine an $F$ that minimizes (2). Criterion 3, a safety-first criterion, requires the same information as criterion 2 plus a value $M$ for maximum acceptable loss. And criterion 4 requires assessment of both $P$ and $u$ to determine an $F$ that maximizes (3). Criterion 2 is tantamount to the special case of criterion 4 for which $u$ has the risk-neutral form $u(g) = -g$, for then maximizing $E(u, F)$ is equivalent to minimizing $E(g, F)$.

The following sections discuss the criteria in turn. A few salient points are summarized here. We say that $F$ is interior if $0 < F < 1$, and extreme if $F \in \{0, 1\}$. 

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Also, \( F^* \) denotes the minimizer of (1). That is, \( F^* \) is the minimax loss fractional coverage.

\( F^* \) is the solution to \( b(F) = 0 \), it is always interior, and it produces the same loss \( a(F^*) \) regardless of \( L \). The minimizer of (2) may always be taken to be extreme, i.e., one should either buy full insurance or no insurance to minimize expected loss. The minimum expected loss never exceeds the minimax loss \( a(F^*) \). When \( A \) is in the neighborhood of $200 million, the minimum expected loss is likely to be less than the minimax loss by several million dollars. However, the actual loss that occurs under criterion 2 can exceed \( a(F^*) \). The safety-first criterion, criterion 3, limits the greatest possible loss and yields an expected loss that exceeds the minimum expected loss but is less than the minimax loss \( a(F^*) \).

We illustrate criterion 4 for maximum expected utility with two types of utility functions for losses. The first is the quadratic utility function

\[
u(g) = -g + cg^2,
\]

with \( 2cA \leq 1 \) so that \( u \) is monotone decreasing for \( g \) from 0 to \( A \). The quadratic form is analytically tractable and demonstrates how the expected-utility maximizing \( F \) responds to \( u \)'s curvature. It requires only estimates of \( E(L) \) and \( E(L^2) \) rather than the full distribution \( P \), which is true also of general utility functions (Meyer, 1987) when \( b \) is always positive. Different responses occur for \( c > 0 \) (convex utility) and \( c < 0 \) (concave utility). Some \( F \) solutions for (3) are extreme, and others are interior, but all with \( c > 0 \) are extreme. There are cases in which \( F^* \) maximizes expected utility, and others in which \( F^* \) minimizes expected utility. Details appear in section 5.

The second utility function used to illustrate the maximum expected utility criterion is the exponential function

\[
u(g) = 1 - e^{hg},
\]

where \( h \) is a scale parameter on the order of \( 1/A \). Larger values of \( h \) place relatively more weight on large losses as compared to small losses. We illustrate the effects of exponential utility on optimal values of \( F \) under several assumptions for the probability-of-loss measure \( P \). Many optimal \( F \) values are extreme, but there are feasible ranges of the parameters for which the optimal coverage is interior. Examples are presented in section 6.

We conclude in section 7 with a brief summary.

2. Minimax loss

We begin with the minimax loss criterion. Because \( a(F) + b(F) \) decreases in \( F \), and \( a(F) \) increases in \( F \), \( \max_L g(F, L) \) in (1) is minimized when \( a(F) + b(F) = a(F) \), i.e., when \( b(F) = 0 \). The solution \( F^* \) to \( b(F) = 0 \) is

\[
F^* = (1 - T)/(1 - T + r),
\]

and the minimax loss value is

\[
g(F^*, L) = a(F^*) = \frac{Ar(1 - T)}{1 - T + rT}.
\]
Because $L$ is absent from $a(F^*)$, the minimax loss is the actual loss for all $L$ when $F^*$ is used.

Figure 1 pictures the situation as $F$ varies from 0 to 1. The upper curves show $\max_L g(F, L)$ and its minimum at $F^*$. The shaded region shows the range of possible losses for different $F$ values.

$F^*$ has a game theory interpretation in which a hostile nature chooses the worst possible $L$ after $F$ has been chosen by the decision maker. The choice of $F = F^*$ protects the decision maker from the worst that nature has to offer by nullifying nature’s power to make matters worse than they might be otherwise.

Viewed in this way, criterion 1 is very conservative. It does limit the maximum possible loss but takes no account of nature’s inherent neutrality. Indeed, because $L$ has no effect on $a(F^*)$, the minimax loss $a(F^*)$ is both the minimum loss and the maximum loss with respect to $L$. As noted in the next section, the minimum expected loss never exceeds $a(F^*)$ and is often substantially less than $a(F^*)$. Figure 1 shows that the range of possible losses shrinks to a point at $F^*$.

3. Minimum expected loss

This criterion assumes that the agency or company can withstand very large losses which are unlikely to occur, and wants to minimize expected loss. The derivative of $E(g, F)$ in (2) with respect to $F$ is

$$\frac{\partial E(g, F)}{\partial F} \bigg|_{E(L)} = A[r - (1 - T + rT)E(L)]/(1 - rF)^2.$$
It follows that $E(g, F)$ increases in $F$ if $E(L) < r/(1 - T + rT)$, and decreases in $F$ if $E(L) > r/(1 - T + rT)$. Hence the $F$ values that minimize expected loss are extreme points, $F = 0$, or 1:

- $F = 0$ if $E(L) < r/(1 - T + rT)$, where $E(g, 0) = A(1 - T)E(L)$;
- $F = 1$ if $E(L) > r/(1 - T + rT)$, where $E(g, 1) = Ar[1 - E(L)]/(1 - r)$.

If $E(L) = r/(1 - T + rT)$, then $\partial E(g, F)/\partial F = 0$ and all values of $F$ have the same expected loss of $a(F^*)$.

Figure 2 pictures the situation as the expected fractional loss $E(L)$ varies from 0 to 1. We see that $\min_F E(g, F)$ increases linearly from 0 at $E(L) = 0$ to the minimax loss value $a(F^*)$ at $E(L) = r/(1 - T + rT)$, then decreases linearly back to 0 at $E(L) = 1$.

The difference $a(F^*) - \min_F E(g, F)$ between the minimax loss and the minimum expected loss as a function of $E(L)$ is shown in figure 3 for $r = 0.2$ and $T = 0.4$. When $A = $200 million, the horizontal line at $A/20$ identifies the two $E(L)$ values at which the difference is $10$ million.

**Figure 2** Minimum expected loss.

**Figure 3** $a(F^*) - \min_F E(g, F)$. 

*Fishburn and Shepp*
4. Safety first

The safety-first criterion minimizes expected loss under the constraint that the maximum possible loss never exceeds a maximum acceptable loss $M$

$$\text{minimize } E(g, F) \text{ subject to } \max_L g(F, L) \leq M.$$ 

Assuming that $a(F^*) \leq M \leq A(1 - T)$, the restriction on $\max_L g(F, L)$ translates into

$$\frac{A(1 - T) - M}{A(1 - T) - rM} \leq F \leq \min\left\{\frac{M}{r(A + M)}, 1\right\},$$

where the lower bound is tantamount to $a(F) + b(F) \leq M$ and the upper bound to $a(F) \leq M$. The bounds are equal when $F$ is the minimax loss fraction $F^*$.

Let $F_1$ and $F_2$ denote the lower and upper bounds respectively when $M = a(F^*)$. Because $E(g, F)$ increases in $F$ when $E(L) < r/(1 - T + rT)$ and decreases in $F$ when $E(L) > r/(1 - T + rT)$, the optimal safety-first $F$ values are

$$F = F_1 \text{ if } E(L) < r/(1 - T + rT), \text{ where }$$

$$E(g, F_1) = \frac{ArF_1 + A[(1 - F_1)(1 - T) - rF_1]E(L)}{1 - rF_1};$$

$$F = F_2 \text{ if } E(L) > r/(1 - T + rT), \text{ where }$$

$$E(g, F_2) = \frac{ArF_2 + A[(1 - F_2)(1 - T) - rF_2]E(L)}{1 - rF_2}.$$

Figure 4 shows the expected loss and the maximum possible loss as functions of $E(L)$ for criteria 1 and 2 and the safety-first criterion when $r = 0.2$, $T = 0.4$, and $M = (1.3)a(F^*)$. With roundoff, $F_1 = 0.6688$ and $F_2 = 0.9330$. We assume that actual losses of $L = 0$ and $L = 1$ are possible.

5. Maximum expected utility with quadratic utility

The maximum expected utility (MEU) criterion accommodates the decision maker’s attitudes towards risk, as well as towards gains or losses of various magnitudes, in risky decision situations. It is widely regarded as the pre-eminent normative decision criterion for such situations (Edwards, ed., 1992; Fishburn, 1970; Raiffa, 1968; Savage, 1954).

The criteria discussed above can be seen as instances of MEU for specific utility functions $u$ that decrease monotonically as the loss amount increases. This was already noted for minimum expected loss, which is equivalent to MEU when $u$ is risk neutral, i.e., when $u(g) = -g$. Allowing $-\infty$ as a utility value, the safety-first criterion with $M \geq a(F^*)$ is a version of MEU for which

$$u(g) = \begin{cases} 
-g & \text{if } g \leq M \\
-\infty & \text{if } g > M.
\end{cases}$$

This extended utility function effectively prohibits losses that exceed $M$. It implies the minimax loss criterion when $M = a(F^*)$. 

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The next section discusses MEU for exponential utility functions. In the rest of this section we examine MEU for the class of quadratic utility functions defined by

$$u(g) = -g + cg^2$$

for $0 \leq g \leq A$, with $c \neq 0$.

We exclude $c = 0$ because it characterizes criterion 2. We also require $2cA \leq 1$ to ensure that $u(g)$ decreases monotonically from $g = 0$ to $g = A$ when $c > 0$. Figure 5 illustrates the quadratic forms for $c > 0$ and $c < 0$. When $c > 0$, $u$ decreases at a decreasing rate and is therefore risk seeking. When $c < 0$, $u$ decreases at an increasing rate and is risk averse or risk avoiding.

Figure 4 Comparison of safety first with criteria 1 and 2.

Figure 5 Curvature for quadratic utility.
Although quadratic utility functions have been used in many studies, they may be less predominant than other simple forms, such as linear, logarithmic, and exponential. Discussions of specific forms for univariate utility functions are included in Bell (1988), Brocket and Golden (1987), Farquhar and Nakamura (1987), Harvey (1981), Savage (1954). We focus on the quadratic forms in this section for three reasons. First, they are analytically tractable in our setting. Second, they approximate many other functions. Third, they highlight utility curvature effects associated with risk-seeking and risk-averse behaviors. In the risk-seeking case, large losses are proportionately less serious than small losses. In the risk-averse case, large losses are proportionately more serious than small losses.

Given $u(g) = -g + cg^2$ let $H(F) = -E(u, F)$, so

$$H(F) = \int_0^1 [g(F, L) - cg(F, L)^2]dP(L).$$

Maximization of $E(u, F)$ is equivalent to minimization of $H(F)$. Let $a = a(F)$ and $b = b(F)$ for convenience. Also let $E(L^2) = \int_0^1 L^2 dP(L)$, and note that

$$E(L^2) - [E(L)]^2 = \int_0^1 [L - E(L)]^2 dP(L) \geq 0.$$ 

We assume that $E(L^2) > [E(L)]^2$, which is true unless $L$ takes some fixed single value almost surely. Integration gives

$$H(F) = a - ca^2 + (b - 2abc)E(L) - cb^2E(L^2).$$

To consider $dH(F)/dF$, let

$$W(F) = [(1 - rF^3)/A]H'(F)$$

so that $W(F)$ has the same sign as $H'(F)$. Differentiation gives

$$W(F) = FX + Y$$

where $X$ and $Y$ are defined below. As further notational conventions, let

$$C = 2cA,$$

$$\alpha = 1 - T,$$

$$\beta = 1 - T + r,$$

$$\gamma = 1 - T + rT.$$

Then

$$X = -r^2(1 + C) + r[(1 + C)\gamma + C\beta]E(L) - C\beta\gamma E(L^2),$$

$$Y = r - [\gamma + Cra]E(L) + C\gamma E(L^2).$$

Note that $W(0) = Y$ and $W(1) = X + Y$, so $Y$ and $X + Y$ tell us the signs of the derivative of $H$ at $F = 0$ and $F = 1$ respectively.

Because $H'(F)$ has exactly one zero, which occurs at $F = -Y/X$, it follows that $H(F)$ has an interior minimum if and only if $Y < 0$ and $X + Y > 0$, as shown in figure 6. When this is true, the minimizing $F$ satisfies $W(F) = 0$, or $F = -Y/X$.

As a corollary, the optimal $F$ is extreme if and only if either $Y \geq 0$ or $X + Y \leq 0$. If $X \geq 0$ and $X + Y \geq 0$, $F = 0$ is optimal; if $Y \leq 0$ and $X + Y \leq 0$, $F = 1$ is optimal; if
$Y > 0$ and $X + Y < 0$, so that $H(F)$ is single-peaked, we need to compare $H(0)$ and $H(1)$ to determine the smaller of the two. Straightforward algebra for $H(F)$ as originally expressed in terms of $a$, $b$, $c$, $E(L)$ and $E(L^2)$ gives

$$H(0) < H(1) \iff E(L)[b(0) - 2ca(1)^2 + a(1)] < a(1)[1 - ca(1)] + cE(L^2)[b(0)^2 - a(1)^2]$$

where $b(0) = A(1 - T)$ and $a(1) = Ar/(1 - r)$ as in figure 1. Generally speaking, we expect to have $b(0) > a(1)$ and, when $c > 0$, $[1 - ca(1)] > 0$. Because $2ca \leq 1$, we must have $1 - ca(1) > 0$ for $c > 0$ whenever $r < 2/3$. We assume in fact that $r < 1/2$ because this simplifies the analysis a little with no real loss of generality. We say more about extreme optima after distinguishing between interior and extreme optima.

Recall that $r/\gamma = r/(1 - T + rT)$ is the dividing point for $E(L)$ between $F = 0$ and $F = 1$ for criterion 2. Our account of interior and extreme optima for MEU uses this point and the sign of $c$ as follows.

The optimal $F$ is interior if and only if

<table>
<thead>
<tr>
<th>$E(L)$</th>
<th>$r/\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; r$</td>
<td>$c &lt; 0$</td>
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<tr>
<td>$c &gt; 0$</td>
<td></td>
</tr>
<tr>
<td>$r - \gamma E(L) &lt; (-C)a[\gamma E(L^2) - rE(L)]$</td>
<td></td>
</tr>
<tr>
<td>$\neq$ NEVER, because $Y \geq 0$</td>
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<tr>
<th>$E(L) = r/\gamma$</th>
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<tr>
<td>$c &lt; 0$</td>
</tr>
<tr>
<td>$c &gt; 0$</td>
</tr>
<tr>
<td>$1 - r(1 + C)[\gamma E(L) - r] &lt; (-C)r[\gamma E(L^2) - rE(L)]$</td>
</tr>
<tr>
<td>$\neq$ NEVER, because $Y \geq 0$</td>
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<tr>
<th>$E(L) &gt; r/\gamma$</th>
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<tr>
<td>$c &lt; 0$</td>
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<tr>
<td>$c &gt; 0$</td>
</tr>
<tr>
<td>$\neq$ NEVER, because $X + Y \leq 0$</td>
</tr>
</tbody>
</table>

We indicate how these occur.

Suppose $E(L) < r/\gamma$, $c > 0$ and $Y < 0$. We show that these yield a contradiction. The hypotheses, $C \leq 1$ and $[E(L)]^2 < E(L^2)$ imply

$$0 < r - \gamma E(L) < Ca[rE(L) - \gamma E(L^2)]$$

$$\leq a_r E(L) - a\gamma E(L^2)$$

$$\leq a_r E(L) - a\gamma[E(L)]^2.$$  

Hence $r - \gamma E(L) < a_r E(L) - a\gamma E(L)^2$, or $r[1 - aE(L)] < \gamma E(L)[1 - aE(L)]$. But $aE(L) < 1$, so $r < \gamma E(L)$, a contradiction. Therefore $E(L) < r/\gamma$ and $c > 0$ imply $Y \geq 0$. 

**Figure 6** Slopes for interior optimum.
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Suppose \( E(L) < r/\gamma \) and \( c < 0 \). The inequality in the table is \( Y < 0 \). The earlier definitions of \( X \) and \( Y \) give

\[
X + Y > 0 \Leftrightarrow [1 - r(1 + C)]\{\gamma E(L) - r\} < Cr[\gamma E(L) - \gamma E(L^2)].
\]

This holds when \( Y < 0 \) because its left side is negative and its right side is positive.

Suppose \( E(L) = r/\gamma \). If \( c < 0 \) then \( Y < 0 \) reduces to \( E(L^2) > E(L)^2 \), which is true, and the same thing happens for \( X + Y > 0 \). Substitution in \( F = -Y/X \) then gives \( F = F^* \). When \( c > 0 \), \( Y < 0 \) reduces to \( E(L^2) < E(L)^2 \), which is false, and in this case \( F^* \) minimizes expected utility. Hence, for the knife-edge case of \( E(L) = r/\gamma \), \( F^* \) is the best decision if \( c < 0 \) and the worst decision if \( c > 0 \) from the MEU perspective. This gives stark proof that the optimal policy strongly depends on the criterion of optimality!

Suppose \( E(L) > r/\gamma \). If \( c > 0 \) then \( X + Y > 0 \) can hold only if \( \gamma E(L^2) < rE(L) \), but this implies \( \gamma E(L^2) < \gamma E(L)^2 \), or \( E(L^2) < E(L)^2 \), which is false. Hence \( c > 0 \Rightarrow X + Y \leq 0 \), and the optimal \( F \) is extreme. When \( c < 0 \), the inequality in the table is \( X + Y > 0 \), which is possible in this case. Moreover, \( Y < 0 \) holds if \( rE(L) \leq \gamma E(L^2) \) because \( r - \gamma E(L) \) is negative. By \( E(L) > r/\gamma \), we have \( [r/E(L)]E(L^2) < \gamma E(L^2) \), and this in conjunction with \( rE(L) < [r/E(L)]E(L)^2 \) (i.e., \( E(L)^2 < E(L^2) \)) yields \( rE(L) < \gamma E(L^2) \). Hence \( c < 0 \Rightarrow Y < 0 \), so \( X + Y > 0 \) is necessary and sufficient for an interior optimum in this case.

An overview of the table shows that there are no interior optima when \( c > 0 \). Given \( c < 0 \) and \( E(L) \neq r/\gamma \), a necessary condition for an interior optimum is \( rE(L) < \gamma E(L^2) \), which must be true if \( E(L) > r/\gamma \) but might be false otherwise. All interior optima satisfy

\[
F = \frac{[\gamma E(L) - r] + Ca[rE(L) - \gamma E(L^2)]}{r(1 + C)[\gamma E(L) - r] + C\beta[rE(L) - \gamma E(L^2)]}.
\]

Differentiation with respect to \( C(= 2c) \) gives

\[
\frac{\partial F}{\partial C} = \frac{\gamma E(L) - r}{X} \left( \frac{\gamma E(L^2) - rE(L)}{X} \right) / X^2.
\]

Moreover, because \( \gamma E(L) - r \geq 0 \) and \( E(L^2) > [E(L)]^2 \), we have

\[
\gamma^2 E(L^2) + r^2 > 2r \gamma E(L).
\]

Suppose there is an interior optimum and \( c < 0 \). If \( E(L) > r/\gamma \), as in the bottom row of the table, then \( \partial F/\partial C > 0 \) reduces to the preceding inequality, so \( F \) increases as \( C \) increases. If \( E(L) < r/\gamma \), as in the first row of the table, \( \partial F/\partial C < 0 \) reduces to the same inequality, so \( F \) decreases as \( C \) increases. Hence, if there is an interior optimum over an interval of negative \( c \) values then, as \( c \) becomes more negative in this interval, optimal \( F \) decreases if \( E(L) > r/\gamma \) and increases if \( E(L) < r/\gamma \).

We take this one step farther to illustrate the behavior of optimal \( F \) when \( c \) becomes very negative. Assume that

\[
\gamma E(L^2) - rE(L) > 0 \quad \text{if } E(L) < r/\gamma,
\]

\[
\gamma E(L^2) - rE(L) > \gamma E(L) - r \quad \text{if } E(L) > r/\gamma.
\]

Then there are interior optima for all large negative \( c \). The preceding equation for \( F = -Y/X \) gives
\[ \lim_{c \to -\infty} F = \frac{F^*}{1 - \frac{r[yE(L) - r]}{r[\gamma E(L^2) - rE(L)]}} = F^{**}. \]

Figure 7 illustrates this and the preceding paragraph.

When there is an extreme optimum, it is uniquely \( F = 0 \) if and only if \( H(0) < H(1) \), i.e.

\[ E(L)[b(0) - 2ca(1)^2 + a(1)] < a(1)[1 - ca(1)] + cE(L^2)[b(0)^2 - a(1)^2], \]

where all terms in brackets are positive, and is uniquely \( F = 1 \) if and only if \( H(0) > H(1) \). Suppose \( c > 0 \). Then the optimal \( F \) will be \( F = 0 \) when \( E(L) \) is small, and it will be \( F = 1 \) when \( E(L) \) is large. There will also be an interior interval of \( E(L) \) values for which there are feasible \( P \) measures that give \( (E(L), E(L^2)) \) pairs at which \( F = 0 \) and \( F = 1 \) are joint optima.

Suppose \( c < 0 \). Let \( d = -c > 0 \). Inequality \( H(0) < H(1) \) is

\[ E(L)[b(0) + 2da(1)^2 + a(1)] + dE(L^2)[b(0)^2 - a(1)^2] < a(1)[1 + da(1)]. \]

When \( c < 0 \) has an extreme optimum, it is \( F = 0 \) when \( E(L) \) is small, and \( F = 1 \) when \( E(L) \) is large. Suppose the assumptions of figure 7 hold and \( E(L) \neq r/\gamma \). The preceding table shows that the optima are extreme when \( d \) is near 0. As \( d \) approaches 0, \( F = 0 \) is optimal if \( E(L)[b(0) + a(1)] < a(1) \), i.e., if \( E(L) < r/\gamma \), and \( F = 1 \) is optimal if \( E(L) > r/\gamma \).

6. Maximum expected utility with exponential utility

Our exponential utility function for losses \( g \geq 0 \) is

\[ u(g) = 1 - e^{hg}, \]

\[ F^* \\
\text{opt. } F \text{ when } E(L) > r/\gamma \]

\[ F** \text{ when } rE(L) > r \]

\[ F^* \\
\text{opt. } F \text{ when } E(L) = r/\gamma \]

\[ F** \text{ when } \gamma E(L) > r \]

\[ F^* \\
\text{opt. } F \text{ when } E(L) < r/\gamma \]

\[ F** \text{ when } \gamma E(L) < r \]

\[ -\infty \leftarrow c \leftarrow 0 \]

\[ 0 \]

\[ 1 \]

\[ 1 \]

\[ \text{Figure 7} \text{ Interior optima when } c < 0. \]
where \( h \) scales losses to reflect the relative severity of large losses compared to small losses. We expect \( h \) to lie in an interval from about \( A/4 \) to \( 2A \), in which case \( hA \in [1/4, 2] \). Larger values of \( h \) put proportionately greater emphasis on large losses as illustrated in figure 8.

With \( u(g) = 1 - e^{hg} \), the expected utility by (3) is

\[
E(u, F) = 1 - \int_0^1 e^{hg(F,L)} dP(L).
\]

It follows that maximization of \( E(u, F) \) with respect to \( F \) is equivalent to minimization of \( K(u, F) = 1 - E(u, F) \)

\[
K(u, F) = e^{ha} \int_0^1 e^{hbl} dP(L)
\]

with \( a = a(F) \) and \( b = b(F) \). We focus henceforth on minimization of \( K \) with respect to \( F \).

Because minimization of \( K \) is less tractable analytically than minimization of \( H \) in the preceding section, we illustrate the behavior of optimal \( F \) for exponential utility under the following three simplifying assumptions for \( P \):

**Assumption 1**: \( P \) is uniform on \([0, 1]\), hence \( P(L \leq x) = x \) and \( dP(L) = dL \).

**Assumption 2**: The venture is either a complete success \((L = 0)\) with probability \( p_0 \) or a complete failure \((L = 1)\) with probability \( p_1 = 1 - p_0 \). Here \( P \) has mass spikes at 0 and 1 and vanishes on \((0, 1)\).

**Assumption 3**: \( P \) has mass spikes of \( p_0 \) at \( L = 0 \) and \( p_1 \) at \( L = 1 \), with \( p_0 + p_1 < 1 \), and is uniform on the open interval \((0, 1)\) with total probability \( q = 1 - p_0 - p_1 \) there.

We comment on each assumption’s effects, but first summarize the general results for typical values of the parameters. When assumption 1 is presumed, full coverage \((F = 1)\) is optimal. Under assumption 2, full coverage is optimal for smaller values of \( p_0 \), zero coverage is best for values of \( p_0 \) near 1, and interior \( F \) are optimal in

![Figure 8 Exponential utility: \( h_1 > h_2 \).](image)
between. When assumption 3 is presumed with realistic values of $p_0$, $p_1$, and $q$, such as 0.8, 0.1, and 0.1 respectively, zero coverage is optimal for ordinary values of $hA$, but if $hA$ is large enough then interior $F$ are optimal but are never greater than the minimax coverage fraction $F^\ast$. Some details follow.

**Assumption 1** When $P$ is uniform

$$K(F) = e^{ha} \left( \frac{e^{hb} - 1}{hb} \right).$$

In particular

$$K(0) = \frac{e^{hA(1-T)} - 1}{hA(1-T)},$$
$$K(F^\ast) = e^{hAr/(1+rT)},$$
$$K(1) = \frac{e^{hAr/(1-r)} - 1}{hAr/(1-r)}.$$

At the extremes we have

$$K(1) < K(0) \iff T < \frac{1-2r}{1-r}.$$  

Because $(1-2r)/(1-r)$ equals $8/9$ at $r=0.1$ and $3/4$ at $r=0.2$, it is very likely that $K(1) < K(0)$.

Assume that $K(1) < K(0)$. We compare the better extreme, $F = 1$, to $F^\ast$

$$K(1) < K(F^\ast) \iff \frac{e^{hAr/(1-r)} - 1}{hAr/(1-r)} < e^{hAr(1-T)/(1+rT)}.$$  

This also is likely to hold for typical values of $r$, $T$, and $hA$. For example, when $r=0.2$ and $T=0.4$, we have $K(1) < K(F^\ast)$ so long as $hA$ is less than about 25.

Additional computations show that $K(F)$ is nearly linear in $F$ for typical parameter values, and that $dK(F)/dF < 0$ at $F = 1$. Consequently, apart from rather unlikely values of the parameters, Assumption 1 implies that full coverage is optimal.

**Assumption 2** When all of $P$'s mass is concentrated at $L = 0$ and $L = 1$, we have

$$K(F) = e^{ha} [p_0 + p_1 e^{hb}].$$

It follows that $dK(F)/dF = 0$ if and only if $e^{hb} = (p_0/p_1)(r/[(1-T)(1-r)])$. Let $R = (p_0/p_1)(r/[(1-T)(1-r)])$. Then, for optimality

$$F = 1 \quad \text{if} \quad \ln R/(hA) \leq -r/(1-r),$$
$$0 < F < 1 \quad \text{if} \quad -r/(1-r) < \ln R/(hA) < 1 - T,$$
$$F = 0 \quad \text{if} \quad 1 - T \leq \ln R/(hA).$$

Figure 9 pictures optimal $F$ as a function of $p_0$ when the other parameters are fixed at $r = 0.2$, $T = 0.4$, and $hA = 1$. We see that full coverage is best if $p_0 \leq 0.6515$, no coverage is optimal if $p_0 \geq 0.8139$, and interior $F$ are optimal otherwise.
Assumption 3  When $P$ has masses $p_0$ and $p_1$ at $L = 0$ and $L = 1$, and is uniform on $(0, 1)$,

$$K(F) = e^{ha} \left[ p_0 + p_1 e^{hb} + q \frac{e^{hb} - 1}{hb} \right],$$

with $p_0 + p_1 + q = 1$. Figure 10 shows how optimal $F$ is affected by changes in $hA$ when the other parameters are fixed at $r = 0.2$, $T = 0.4$, and $(p_0, p_1, q) = (0.8, 0.1, 0.1)$. Zero coverage is optimal for $hA$ less than about 1.6. Then, as large losses take on proportionately greater significance, the optimal $F$ increases away from 0 but never rises above $F^* = 0.75$.

A similar picture holds for somewhat different values of the other parameters. In particular, $K(F^*) < K(F)$ for all $F > F^*$ and, for each $F < F^*$, sufficiently large values of $hA$ imply $K(F) > K(F^*)$. As $hA \to \infty$, the optimal fractional coverage approaches $F^*$ from below. This is consistent with the interpretation of the minimax loss criterion in the second paragraph of the preceding section.

**Figure 9** Optimal $F$ versus $p_0$ for Assumption 2.

**Figure 10** Optimal $F$ versus $hA$ for Assumption 3.
7. Discussion

This paper was motivated by the possibility of using fractional coverage in an insurance strategy to control financial risk in ventures with huge potential losses, illustrated by communication satellite missions which typically have about a 15 per cent chance of complete or partial failure, and an 85 per cent chance of successful launch and mission completion. Our aim has been to outline a menu of criteria for choosing the best fractional coverage $F$ and to illustrate their implications. Some optimal $F$ values are extreme, either $F = 0$ for no coverage or $F = 1$ for full coverage, whereas other cases indicate an optimal $F$ strictly between 0 and 1. We refer to the latter as interior solutions or optima.

The first three criteria, minimax loss, minimum expected loss, and loss-limited minimum expected loss, are easily explained without utility theory. They require minimal information about loss probabilities because the loss function is linear in fractional loss $L$. The minimax loss solution $F^*$ is always interior and is quite conservative, whereas the minimum expected loss solution is always extreme with an expected loss less than the minimax loss. The latter inequality is true also of the loss-limited (safety-first) minimum expected loss solution, which can be either extreme or interior. In comparison with unconstrained minimum expected loss, the safety-first solution generally has a smaller maximum possible loss but a larger expected loss.

Each of the first three criteria can be interpreted as a maximum expected utility criterion provided that $-\infty$ is admitted as a loss utility. More traditional utility functions were used to illustrate maximum expected utility solutions for various parameter configurations. We looked first at quadratic utility and then considered exponential utility. In each case, the solution could be either extreme or fractional and, when the utility of loss has a large negative curvature, can approach $F^*$ (exponential) or $F^{**}$ (quadratic) as defined for figure 7.

Because circumstances or risky ventures with large potential losses can differ radically, we have not suggested a specific solution criterion for all situations. The menu of criteria described in the paper is designed to aid in making reasoned choices of insurance coverage in situations of the type that motivated our inquiry.

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References

Fractional insurance


