Stat101
Spring 2005

Lecture 9

Topics

- Probability models
- Parallel universes
- Sets and Venn Diagrams
- Events
- Assigning probabilities

Emphasis

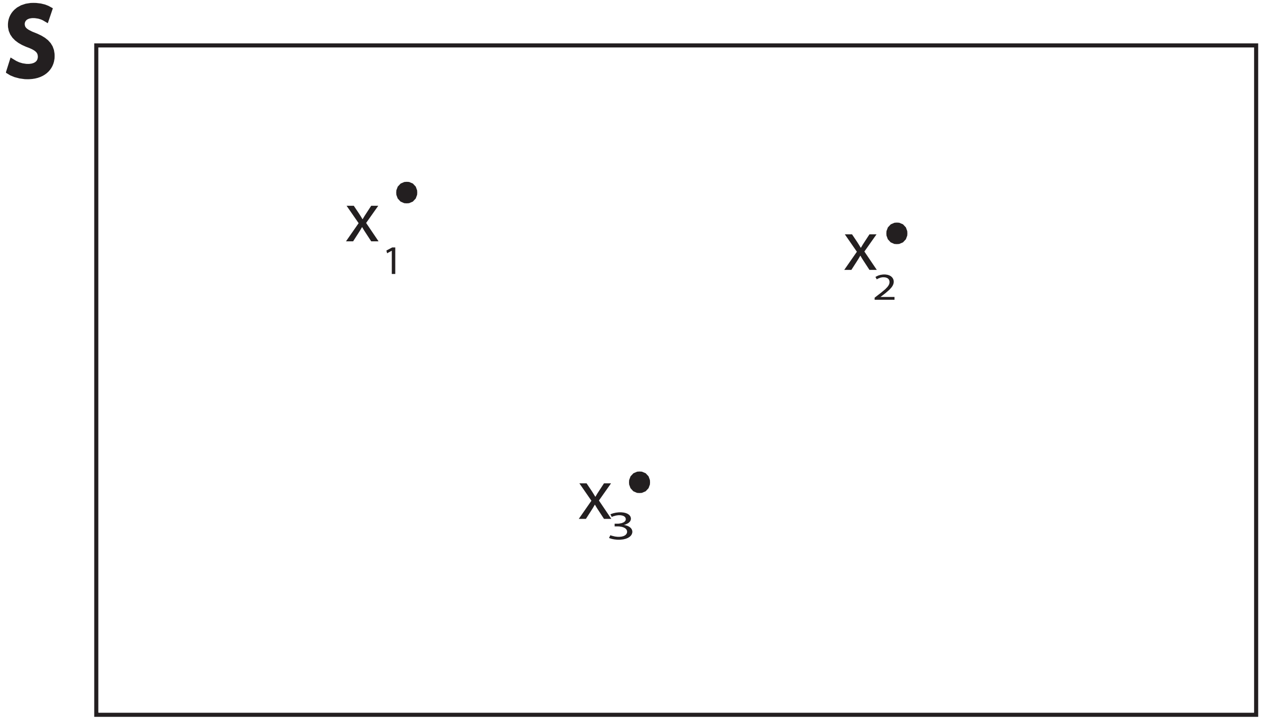
So not to lose focus we stress our (very practical) intentions:

- Applications to statistics, particularly random fluctuations
- Portfolio analysis
- Decision making

Building a probability model: step 1

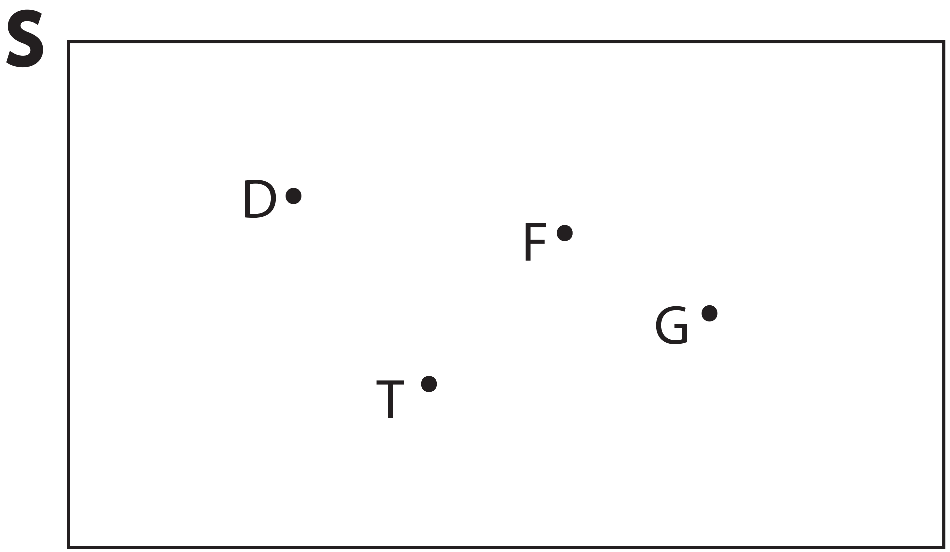
- Create a parallel universe, i.e. specify what is possible
- This is called a *sample space* and denoted as $\mathcal{S}$
- A sample space $\mathcal{S}$ contains *sample points* or *atoms*
- They may be finite or infinite. If finite we may write $\mathcal{S} = \{x_1, x_2, \ldots, x_n\}$, so in this case $\#\mathcal{S} = n$.

- We often use a rectangle to represent the sample space:



Abba's oil well example

- This is going to be our main toy in what follows so we may as well introduce it now.
- Suppose an oil well is classified into one of 4 (uncertain states) states: dry, trickler, flower or gusher or D,T,F,G for short. We write $\mathcal{S} = \{D, T, F, G\}$
- Using a Venn Diagram we write

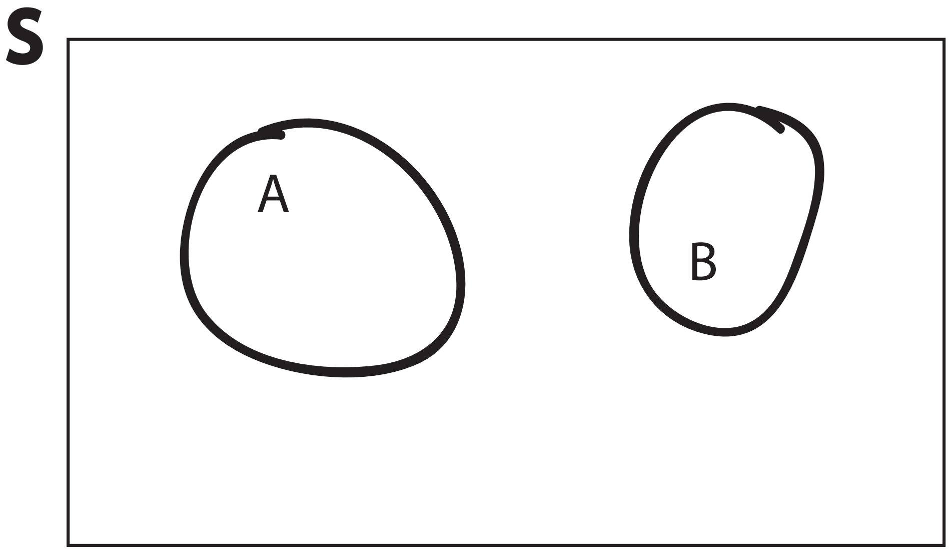


Some more examples

- Roll a die, then $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$
- $\mathcal{S}$=range of real numbers
- $\mathcal{S}$=roll two dice. What is $\#\mathcal{S}$?
- $\mathcal{S}$=flip 500 fair coins. What is $\#\mathcal{S}$?

Event

- An *event* is the thing that we are interested in.
- In the language of sets an event A is simply a subset of the sample space S
- Using Venn diagrams:



Examples of events

- $A = \{F, G\}$
- Roll 2 dice, A is event score is 7 so $A = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$
- Play Poker, A could be the event that one is dealt a flush.

Some special events

We single out some special events:

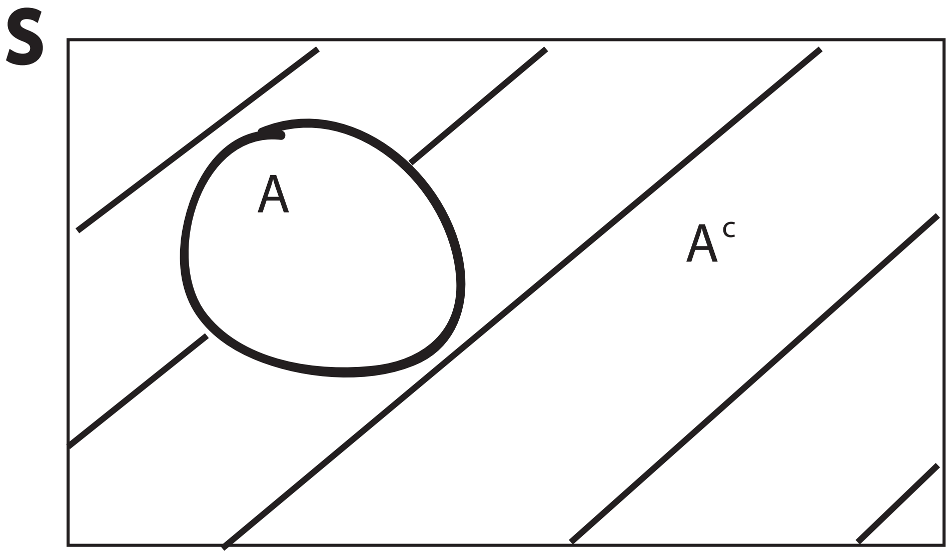
- $\emptyset$ *null* or *impossible* event
- $\mathcal{S}$ *sure* event (something in this sample space will happen because we defined it so)
- $\{x\}$ *elementary* events; these contain just one sample point.

Step 3: algebra of events

Often we want to combine events, and may be interested in, for example, the occurrence of one event or another occurring, or two events occurring simultaneously. We express this simple operations using the language of set theory. Suppose that A and B are events. So now we discuss rules for combining events.

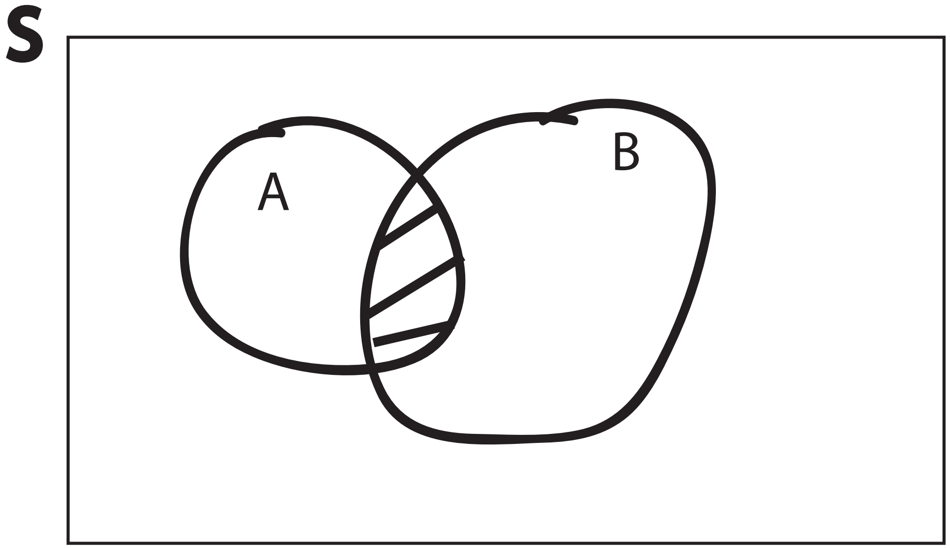
Complement

- A^c, A' or $\bar{A}$ is called the *complement* of A, and contains those points in the sample space $\mathcal{S}$ not in A.
- Using Venn diagrams:



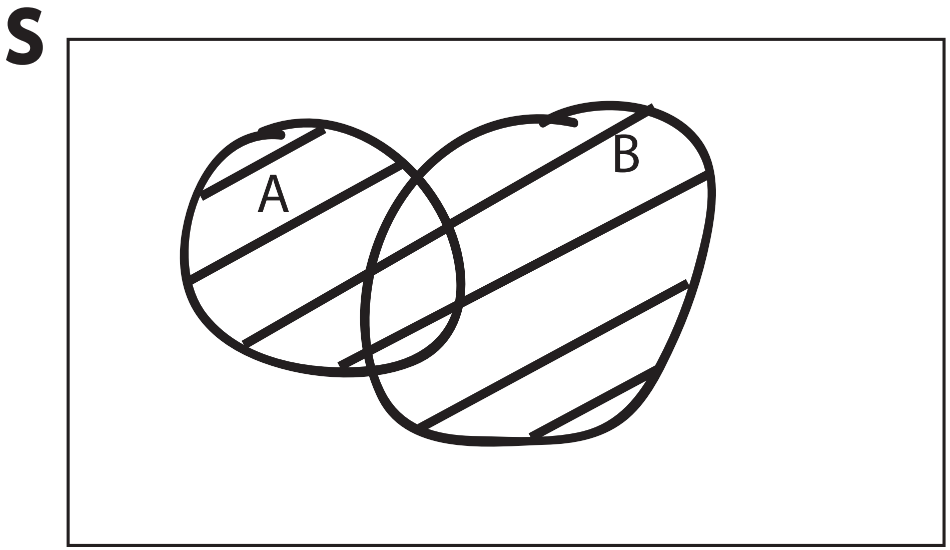
Intersection

- $A \cap B$ is called the *intersection* of A and B and contains those sample points common to both events.
- We have



Union

- $A \cup B$ is called the *union* of A and B and contains those sample points which are either in A or B or both.
- The picture is:



Examples

- Roll a die so that $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$
- Let A, B, C, D and E be the following events: $A = \{1, 3, 5\}$, $B = \{3, 6\}$, $C = \{1, 5\}$, $D = \{2, 4\}$, $E = \{6\}$.
- What are A^c, $A \cap B$, $A \cup B$?

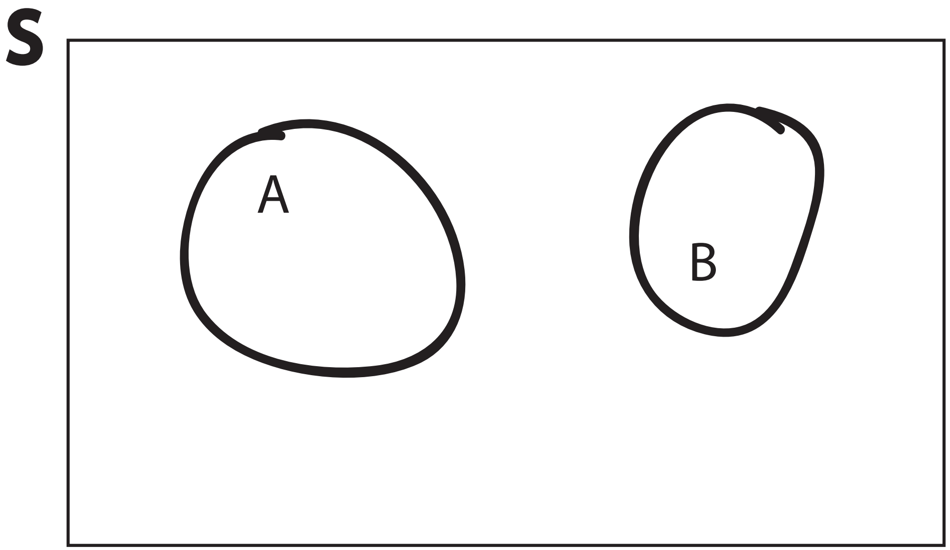
An piece of wisdom

- A^c is often easier to work out than A.
- In the example above, let A be the event 'at most 5', then $A = \{1, 2, 3, 4, 5\}$ but $A^c =$?
- Throw two dice. Let A be the event throw at least one 6. Then A^c is the event…

Relationships between events

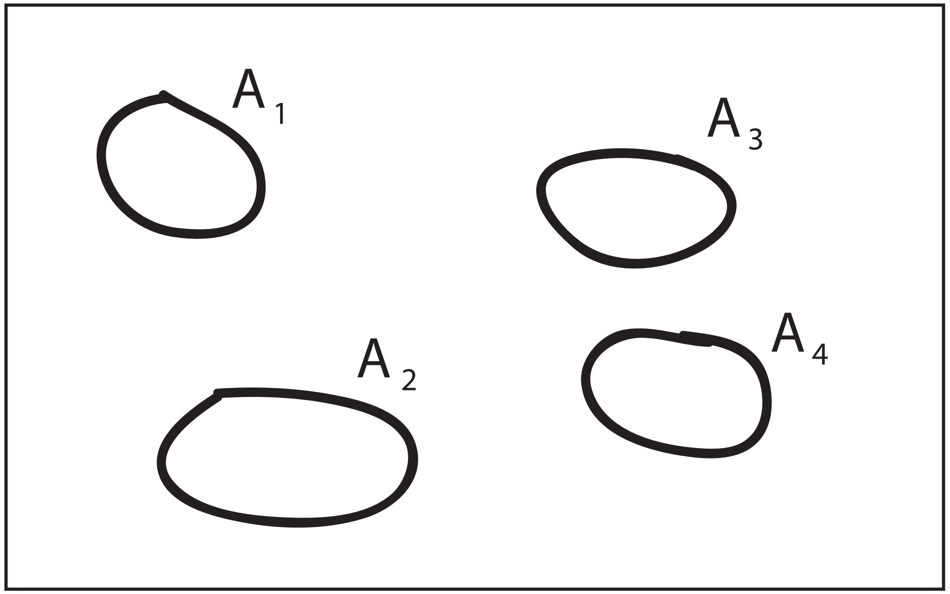
It is exceedingly convenient (but dull) to introduce some terms to describe relationships between events.

- A and B are said to be *disjoint* if they have no points in common: $A \cap B = \emptyset$
- In pictures:



Mutually exclusive

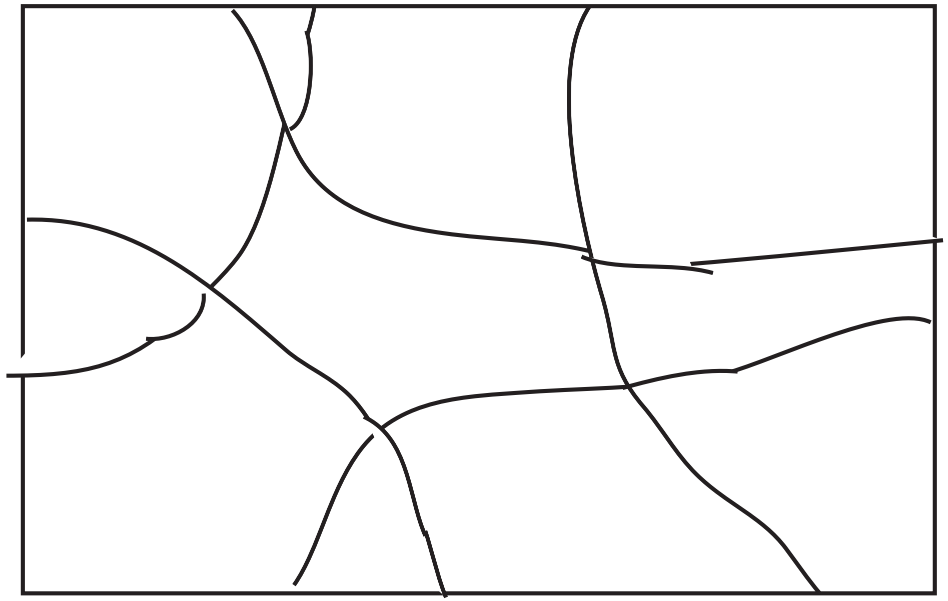
- A collection of events $A_1, A_2, \ldots A_n$ are said to be *mutually exclusive* if they are *pairwise disjoint* $A_i \cap A_j = \emptyset$
- In pictures:



- B, C, and D

Collectively exhaustive

- A collection of events $A_1, A_2, \ldots A_n$ are said to be *collectively exhaustive* if they cover $\mathcal{S}$: $\cup A_i = \mathcal{S}$
- In pictures:



- A, B, C, D and E.

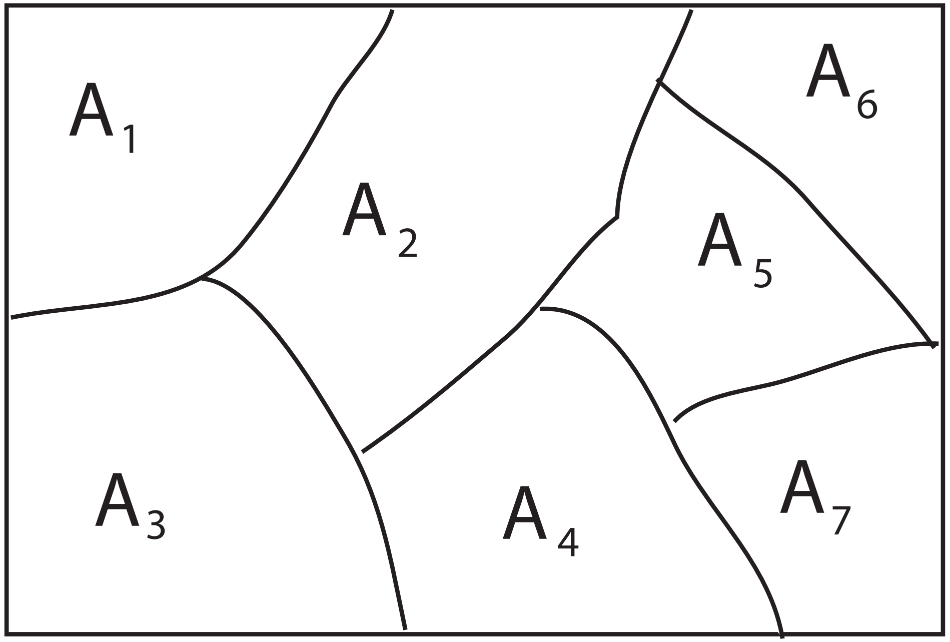
Partition

This one is the really important one

- Sometimes called a *map*
- A collection of events $A_1, A_2, \ldots A_n$ are said to be *partition* of the sample space $\mathcal{S}$ if they are exhaustive *and* mutually exclusive: $\cup A_j = \mathcal{S}$ and $A_i \cap A_j = \emptyset$
- A, D and E

Partition (cont)

- In pictures:



- No matter what occurs it can only occur once
- Discussion: contingency tables.

Nota Bene

We will use all these concepts in the next classes when we do conditional probability, particularly Bayes' Rule.

Step 4: assigning probabilities

- Probability, in the broad, is a measure of likelihood, or to be more poetic, chance.
- How do we assign a measure of chance to an event?
- There are a number of ways to skin this cat.

The old-fashioned way

- Sometimes called the classical approach
- Intuition: every outcome has equal probability (i.e. ‘fair’)
- Let $|A| = \#A$. Then

$$\text{prob}(A) = \frac{|A|}{|\mathcal{S}|}$$

- This definition has a basic role in statistics, e.g. sample surveys.

The frequentist way

- Assume we can replicate the problem of interest many times (at least conceptually), e.g. coin tossing experiments
- Then

$$\begin{aligned}\text{prob}(A) &= \text{ fraction of times that } A \text{ occurs} \\ &= \frac{\#A \text{ occurs}}{\#\text{trials}}\end{aligned}$$

The subjective or Bayesian approach

- Best done in the context of an example
- Flip a fair coin and suppose if it falls on heads you win \$ 1, otherwise you lose \$1 which goes to Paul.
- What is a fair price for this game?
- Suppose it's worth $\$p$ to you. It's also worth that to Paul.
- Hence $\$2p = \$1$, and so $p = \frac{1}{2}$ and this is the probability that the coin falls heads up.

Some more remarks on the Bayesian approach

- Clearly, in general, $0 \leq p \leq 1$
- If you are definitely given nothing $p = 0$ and if you are definitely given $\$1$ then $p = 1$.
- Probabilities are based entirely on personal opinion.

The case of the Russian bond

- Consider a bond for 1 million roubles offered for 1 rouble
- Seller's view: chance of bond being bought back at par is less than 1 in a million
- Buyer's perspective: chance is more than 1 in a million (otherwise you'd reduce your offer)
- Therefore $p = 10^{-6}$ is the probability of it's redemption.

Footnote

- These are hard concepts and take some getting used to
- Read sections 4.1 and 4.2 in your book, and if you have the stomach take a look at 5.4.