Finite Horizon Dynamic Programming: Getting Value from Spending Symmetry

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Some History and Motivation

- Famous combinatorial problems with long mathematical history on sequences of n real numbers, or permutations of the integers $1, \ldots, n$
  - Erdős and Szekeres (1935): monotone subsequences
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  - Longest monotone subsequences: Hammersley (1972), Kingman (1973), Logan and Shepp (1977), Veršik and Kerov (1977), . . .
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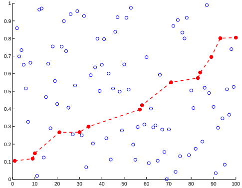
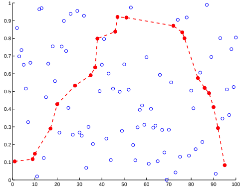
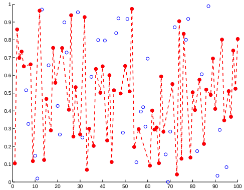
  - Longest Alternating subsequences: Widom (2006), Pemantle (cf. Stanley, 2007), Stanley (2008), Houdré and Restrepo (2010)







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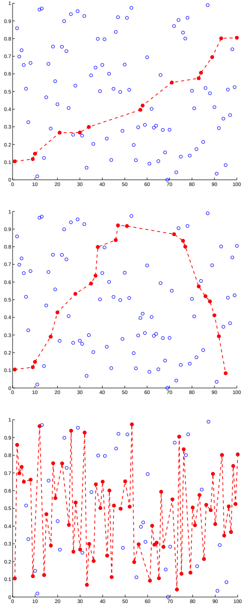
- Now ... Study the sequential (on-line) version of these problems
  - Objective: maximize the expected length (number of selections) of monotone, unimodal and alternating subsequences



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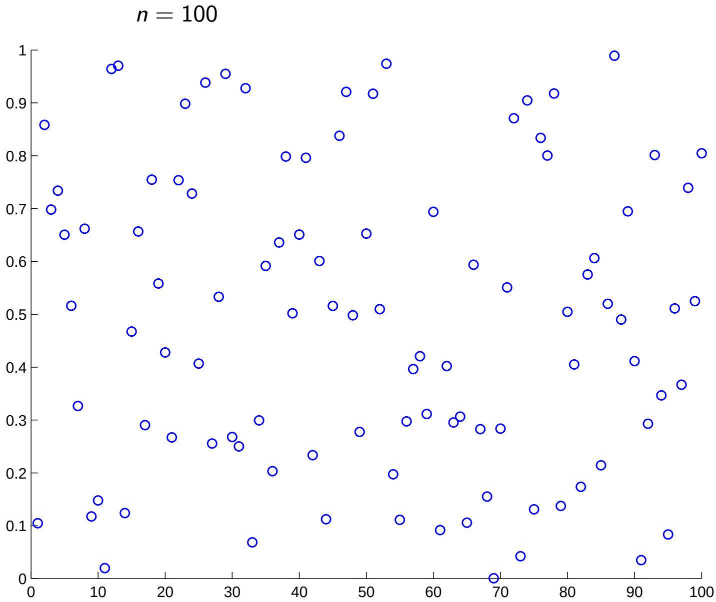

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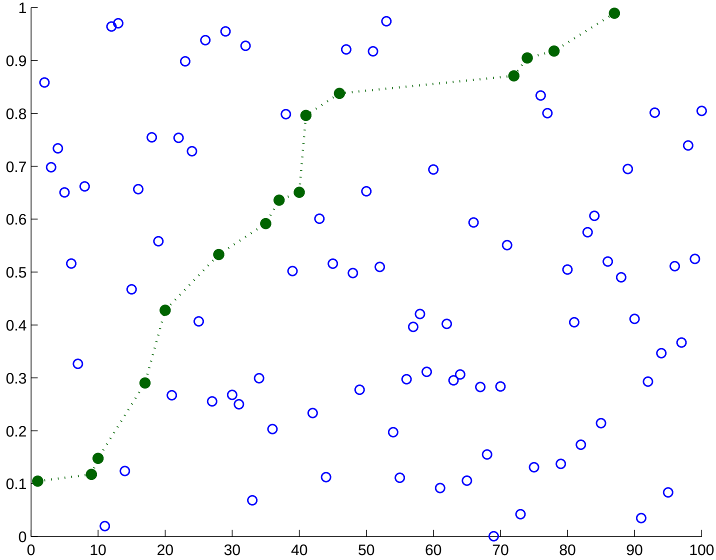

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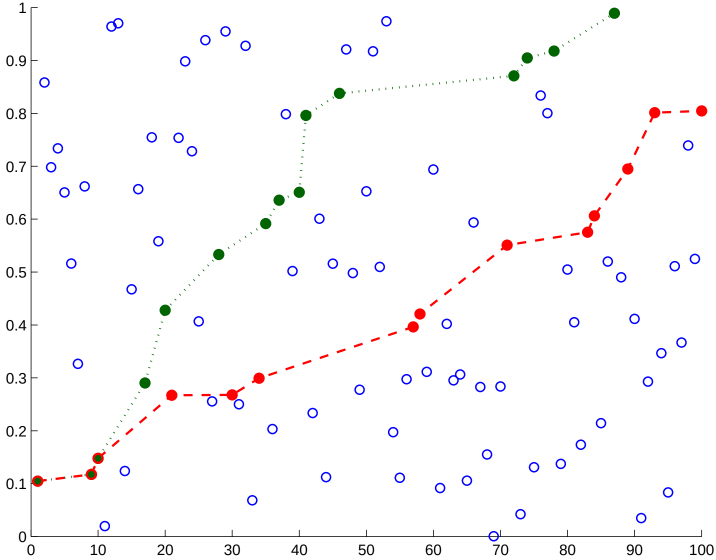

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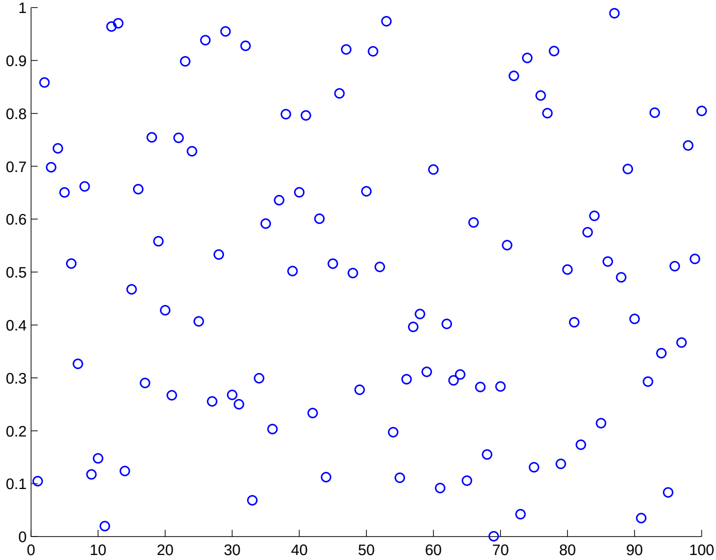

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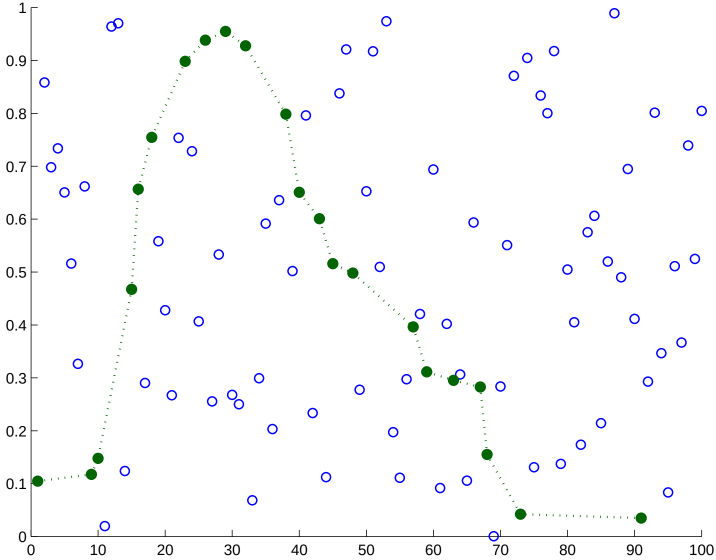

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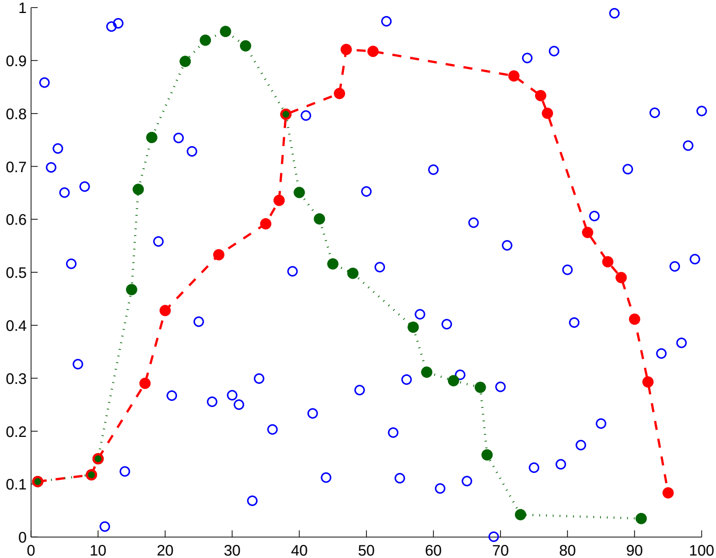

Summary View of Means in Some On-Line Selection Problems

- How Much Better Does a "Prophet" Do Asymptotically?

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- There is a CLT for the On-Line Alternating Subsequence Problem (briefly noted in next frame)
- There has much further work on the *On-Line Selection of a Monotone Increasing Subsequence*, the original motivating problem. This will get most of our attention.

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Theorem (Arlotto & Steele, AAP 2014)

There is a constant $\sigma > 0$ *such that*

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- Source of Juice? Very detailed analytical understanding of the acceptance threshold functions.

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There is a policy $\pi^* \in \Pi(n)$ *such that*

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- Puzzle: A CLT is far from a sure thing. For the off-line problem one does NOT have a CLT — One has the famous Tracy-Widom Law.

Poissonization: A Homogenizing Trick with Benefits

- If one takes a sample size $N(t)$ that is Poisson with mean t there are several benefits: (a) optimal policies are stationary — no horizon effects and (b) one gets the machinery of infinitesimal generators, Dynkin Martingale, etc. There is long history of applications, perhaps starting with Lucien LeCam.

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  - De-Poissonization of a *Decision Problem* is a whole new kettle of fish.
  - Only "one of the five steps" to the proof of the CLT for the finite horizon LIS uses what one could call classical de-Poissonization.

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- Only one of these steps has what one can properly call a de-Poissonization.

Out of Five: Only One for Free

- The CLT of Bruss and Delbean has five parts:
  1. Mean lower bound: $(2t)^{1/2} - O(\log(t))$
  2. Mean upper bound: $(2t)^{1/2}$
  3. Variance lower bound: $\frac{1}{3}(2t)^{1/2} - O(1)$
  4. Variance upper bound: $\frac{1}{3}(2t)^{1/2} + O(\log t)$
  5. The CLT itself
- Only one of these steps has what one can properly call a de-Poissonization.
- De-Poissonization gives us the mean lower bound for the finite horizon problem — and leaves us four steps to go.

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- But we have another property: the map $\phi(j) = \mathbb{E}[L^o_j]$ is concave. Jensen's inequality then forks up

$$\mathbb{E}[L^o_{N(n)}] \leq \sum_{j=0}^{\infty} e^{-n} \frac{n^j}{j!} \mathbb{E}[L^o_j] \leq \mathbb{E}[L^o_n].$$

Thus, we have *lossless transference* of any mean lower bound from the Poisson model to the Finite Horizon model.

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- How about the upper bound for $\mathrm{Var}[L_n^o]$?
- Alessandro and I were stuck here for a long time.

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- The flood gate is opened and more analysis of the same flavor (but with plenty of details) lead us through the Martingale CLT to a CLT for the Finite Horizon Selection Problem for LIS:

$$\frac{3^{1/2}\{L_n^o - (2n)^{1/2}\}}{(2n)^{1/4}} \Longrightarrow N(0,1).$$

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- ¡Gracias por su atención !

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