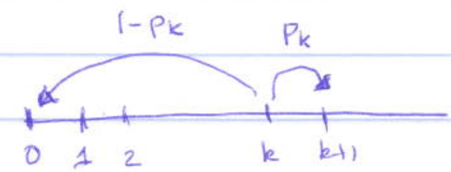
Example of a Chain without a stationary Measure

Following the suggestion of James Norris we consider the following chain:



$$p(0,1) = 1 = p_0$$

$$p(k,k+1) = p_k \qquad k \ge 1$$

$$p(k,0) = 1-p_k \qquad k \ge 0$$

The Equation of stationarity:

$$(1a) \qquad \mu(0) = q_1\mu(1) + q_2\mu(2) + q_3\mu(3) + \cdots$$

$$(1b) \qquad \mu(k) = p_{k-1}\,\mu(k-1) \qquad k \ge 1$$

From (1b) $\quad \mu(k) = p_{k-1}p_{k-2}\cdots p_0\,\mu(0) \qquad k \ge 1$

So by (1a)

$$(*) \qquad \mu(0) = \mu(0)\sum_{k=1}^{\infty} p_0p_1\cdots p_{k-1}\,q_k$$

Now notice

$$\sum_{k=1}^{n} p_0p_1\cdots p_{k-1}q_k = \sum_{k=1}^{n}(p_0\cdots p_{k-1} - p_0\cdots p_k)$$

$$= p_0 - p_0\cdots p_n = 1 - p_0p_1\cdots p_n$$

Let $p = \lim_{n\to\infty} p_0p_1\cdots p_n$

$$\sum_{k=1}^{\infty} p_0\cdots p_{k-1}q_k = 1-p$$

so by $(*)$

$$(**) \qquad \mu(0) = \mu(0)(1-p)$$

(CONT.)

Now we make a specific choice of the p_K.

Take $p_0 = 1$ and $p_k = e^{-\alpha_k}$

where $\alpha_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ for $k \geq 1$

Note $\sum_{k=1}^{\infty} \alpha_k = 1$ so $\prod_{k=0}^{\infty} p_k = e^{-1}$

By $(**)$ we have

$$\mu(0) = \mu(0)(1-e^{-1})$$

so

$$\mu(0) = 0$$

By (1b) this gives

$$\mu(k) = 0 \quad \text{for } k = 1, 2 \ldots$$

Thus the only measure satisfying (1a) and (1b)

is the "trival measure" $\mu(k) \equiv 0$ for all $k \in S$