Stochastic Calculus and Financial Applications
Final Take Home Exam (Steele: Fall 2008)

Instructions. You may consult any books or articles that you find useful. If you use a result that is not from our text, attach a copy of the relevant pages from your source. You may use any software, including the internet, Mathematica, Maple, R, S-Plus, MatLab, etc. **Attach any Mathematica (or similar) code that you use.**

- You may **NOT** consult with any other person about these problems. If you have a question, even one that is just about the meaning of a question, please contact me directly rather than consult with a fellow student.
- I may post "bug reports" or clarifications on our web page, and you should regularly check for these.
- You should strive to make your answers as **clear and complete** as possible. Neatness counts — especially of thought, but even of handwriting. If I can't read it, I can't grade it.
- Never, ever, write down anything that you know — or even vaguely suspect — to be false. If you understand that your argument is incomplete or only heuristic, this may be fine, but it should be properly labeled as incomplete or heuristic.
- Don't skip steps. If I can't go from line n to line $n+1$ in my head, something is missing. If you use Mathematica or a fact from a table, please say so **and document it.** Otherwise, I stare and stare at line n wondering how you got to $n+1$ in your head while I can't.
- Use anything from anyplace, but do not steal. If you make use of an argument from some source, give credit to the source. If you find the complete (and correct!) solution to a problem in a book or on the internet, just print out the pages and attach them. You will get full credit.
- Write on only one side of a page. Use decent, homogeneous, high quality paper. No dinner napkins, hoagie sacks, *Indian Chief* Tablets, etc.
- Begin each new solution on a new page.
- Arrange your solutions in the natural numerical order. If you do not do problem K, then include a self-standing page that says "Problem K was not done."
- Staple your pages neatly with a high-quality stapler with appropriate length and weight to do a clean and secure job.
- **As discussed in class, you MUST use and complete the cover page given at the website. Self-evaluation is hugely valuable.**

GENERAL ADVICE

1. In your solutions, please do not just write down things that you think are relevant even though they do not add up to an honest solution. Such lists are useful when you are working on a problem, but if you offer list as a solution your keeping yourself from having the "missing" idea.

2. If you can explain **clearly** something that you tried that did not work, this sometimes is worth a few points. Please do not abuse this offer. With experience, one learns that many sensible ideas do not work. Almost by definition, this is what separates the trivial from the non-trivial.

3. Try to keep in mind that a good problem requires that one "overcome some objection." What distinguishes a problem from an exercise is that in a good problem a routine plan does not work. The whole point is to go past the place where routine ideas take you. Still, don't shy away from the obvious; **almost all of the "problems" here are "exercises."**

4. If you do something **extra** that is valid, you can get "bonus" points. These special rewards cannot be determined in advance. They are usual small, but they can be substantial — and they do add up.

5. The most common source of bonus points is for saying something particularly well. Clear, well-organize, solutions are gems. They deserve to be acknowledged.

THE BIG PICTURE

Almost certainly these instructions will seem to be overly detailed to you. It is true that they are detailed, but they evolved case by case. Each rule deals with some previous misunderstanding. When you start teaching (and grading) I encourage you to follow this example. There is no harm in a having a few creative (yet compassionate) eccentricities.

There is a final motivation for this long list of rules and suggestions. **Detailed instructions provide clear coaching for excellence.** We all do wonderfully better when we are lucky enough to know what we need to do. This is the kind of break one seldom gets in research.

Due Date and Place: The exam **with its completed self-evaluation cover sheet** is due in my mail box in JMHH Suite 400 on in my office JMHH 447 on the date given on our web page. Solutions may be emailed or sent by FedEx. Fax is not acceptable.

Problem 1: A Warm-Up Exercise

Suppose that Z is normally distributed with mean zero and variance one. Derive the pretty formula

$$E[\max(0, a + bZ)] = a\Phi(\frac{a}{b}) + b\phi(\frac{a}{b}) \quad \text{for all } a \in \mathbb{R} \text{ and b } \in \mathbb{R}^+.$$

Here, of course, Φ and ϕ are just what you think they are. Exam taking hint: When there is an "isolated" warm-up fact, it's probably going to be handy later in the exam!

Problem 2: A Left Over UI Property

Consider BM with drift, $X_t = \mu t + \sigma B_t$ where $\mu > 0$. For $A > 0$ let $\tau = \min\{t : X_t = A\}$. We know already that $P(\tau < \infty) = 1$. Earlier we used without proof the fact that the collection of random variables

$$\{X_{t\wedge\tau} \,:\, 0 \leq t < \infty\}$$

is uniformly integrable. Prove this by showing the stronger fact that for all $1 \leq p < \infty$ we have

$$E(\sup_{0\leq t<\infty} |X_{t\wedge\tau}|^p) < \infty.$$

Problem 3: A Moment for Reflection

Consider the Ornstein-Uhlenbeck process

$$dX_t = -\alpha X_t \, dt + \sigma \, dB_t \quad \text{with } X_0 = x > 0,$$

where as usual we take $\alpha > 0$ and $\sigma > 0$. First, recall that we know the distribution of X_t for each fixed $t \geq 0$. Now let $\tau = \min\{t : X_t = 0\}$ and — to get started — show that $P(\tau < \infty) = 1$. Finally, calculate the distribution function $F(t) = P(\tau \leq t)$. Your answer should be given explicitly in terms of the Gaussian distribution function.

Hint: You should use some analog of the reflection principle that we used to study the distribution of the supremum of BM. A full justification of even the usual reflection principle requires the so-called strong Markov property, which we glossed over, so you cannot really give a 100% rigorous proof of the reflection principle that you'll use here. Instead, you should justify your new reflection principle with an argument that would please a physicist or engineer.

Problem 4: One More SDE Solution Method

Consider the SDE

$$dX_t = \alpha\, dt + \sigma X_t\, dB_t \quad \text{with } X_0 = 0.$$

This is the equation that one would get by incorrectly recalling the Ornstein-Uhlenbeck SDE. Here the location of the X_t factor has been switched from the drift to the volatility. Solve this SDE! Your answer should be an explicit functional of the Brownian motion path.

Note: I'd like to give this without any hint, and it is certainly too big a hint to suggest the idea of multiplying the SDE by $Y_t = \exp(-\sigma B_t + \frac{1}{2}\sigma^2 t)$. Still, without some hint you might waste a lot of time trying our old methods just to see them all fail. Moreover, since this "multiplier method" is about the last of the semi-general SDE solution methods, it really does need to be mentioned.

Problem 5: PDE for Second Moments

Suppose that X_t satisfies the Markovian SDE

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dB_t \quad \text{with} \quad X_0 = x \in [a, b]$$

where $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}^+$ are suitably well-behaved functions.

(a) Show that if the function V_1 satisfies the equation

$$\mu(x)V_1'(x) + \frac{1}{2}\sigma^2(x)V_1''(x) = -1$$

and the boundary conditions $V_1(a) = 0$ and $V_1(b) = 0$, then

$$V_1(x) = E[\tau \mid X_0 = x] \quad \text{where } \tau = \min\{t : X_t = a \text{ or } X_t = b\}.$$

(b) Next, show that if V_2 satisfies the equations

$$\mu(x)V_2'(x) + \frac{1}{2}\sigma^2(x)V_2''(x) = -2V_1(x)$$

and the boundary conditions $V_2(a) = 0$ and $V_2(b) = 0$ then

$$V_2(x) = E[\tau^2 \mid X_0 = x].$$

(c) Finally, solve these ODEs for V_1 and V_2 in the case of Brownian motion where $\mu(x) \equiv 0$ and $\sigma(x) \equiv 1$. Subject your solutions to the "looking back" process that is discussed in the text.

Problem 6: Another Explicit SDE

Consider the vaguely implausible SDE

$$dX_t = \{1 - \log(X_t)\}X_t dt + \sigma X_t dB_t \quad \text{with } X_0 = x > 0.$$

Here, you might rightly worry if this SDE has a solution since the dt coefficient is not Lipschitz. Still, it does have a solution, and, moreover, the solution can be expressed as deterministic function of a stochastic integral of a deterministic function. Find this solution.

Again, I hate to offer any hints, but sometimes in situations like this it is useful to look for an equation for some function of X_t. For example, you may (or may not) find it useful to look for a linear SDE for $Y_t = \log(X_t)$ or something similar.

Problem 7: A Structured Product

Imagine yourself designing a retail product that gives the customer some participation in the SP500 index (the "market") but which guarantees that the customer will not lose money. In particular consider a product of the form:

- It sells for \$10.
- It pays no dividends (or interest).
- At the end of one year it pays the \$10 back if the market is down for the year.
- If the market is up at the end of the year, the contract pays \$10 plus p times \$10 times the percentage increase in the market. Here, of course, we have to take $0 < p < 1$ or we have given too good a deal.

In Black-Scholes world, what would p be to make this contract arbitrage free? The seller of this contract could then subtract a little bit (or a lot!) for his time and trouble.

Work out the numerical value of p under some natural assumptions such as $r = .05$, μ unknown, $\sigma = .30$ and other choices of interest. Finally, what would p be if the contract just guaranteed that the customer would not lose more than 5%?

Here are some considerations:

- Are you really sure that μ really is irrelevant here? No customer would ever believe that!
- You can also assume the customer will ignore any counter party risks. After all, what is the chance that a big investment bank could go broke?
- You may want to synthesize this contract with other kinds of instruments.

- As a simplifying assumption (and an embedded hint), you may ignore any the distinction between European and American style contracts. This is a rough but reasonable approximation under normal circumstances (if those are ever relevant again). You should comment on the direction that you would expect p to move if you did not make this approximation.

Problem 8: European Stock Trading Option

Consider a Black-Scholes world where the bond satisfies $d\beta_t = r\beta_t dt$ for a constant r and where there are two stock asset with prices S_t^A and S_t^B that satisfy the SDEs

$$dS_t^A = \mu_A S_t^A dt + \sigma_A S_t^A dB_t \quad \text{and} \quad dS_t^B = \mu_A S_t^B dt + \sigma_B S_t^B d\tilde{B}_t$$

where the two Brownian motions are independent.

Think of yourself as the owner of one A share. Some guy proposes to write you an option that will give you the right, but not the obligation, to exchange your one A for one B share at time T. What is the time t arbitrage free price of this option. You should work this out as completely as you can. In particular, you should get a formula that "looks like" the Black-Scholes formula. Here are some features to expect and some issues to ponder:

- We have to expect that μ_A and μ_B will disappear.
- We have to use what we have really proved, and we have proved nothing about models with three assets!
- We could use a PDE approach or a Harrision-Kreps approach. Both work. HK looks seems easier to me.

Problem 9: Options in A Strange Model

Consider an asset that has a price that is given by the model

$$S_t = S_0 + \mu t + \sigma B_t.$$

This is certainly infeasible for a stock price since it can go negative, but it may be a crude first-pass model for a futures contract. In that case one would even have $S_0 = 0$. Derive the time zero arbitrage free value of an European call option on this asset with strike price K and expiration time T. Here you may assume that the interest rate is zero. The answer is very nice.

- Don't forget to check that the model is complete.
- Consider the at-the-money option (i.e. $K = S_0$). Does the dependence on σ and T make sense? Are there constants in the formula that are perhaps surprising to see in a financial context?

PROBLEM 10: INTUITIVE FEATURES OF CALL OPTION PRICES

The purpose of this problem is to develop additional intuition about call options under the general Markovian stock model

$$dS_t = \mu(S_t)S_t dt + \sigma(S_t)S_t dB_t$$

and the constant interest rate bond model, $d\beta_t = r\beta_t dt$. We also assume that the drift and volatility are nice enough to make this model complete on the time interval $[0, T]$. Accordingly, the Harrision-Kreps formula is assumed to be valid.

Each of the assertions below is true under the classic Black-Scholes model where μ and σ are constant. Your task is to determine which are still true under the more general model. You should then prove the true ones and give counterexamples to the false ones (if any). Here $C(S, t; K, T)$ denotes the arbitrage-free price of a European call at time t with stock price S at time at time t, strike price K, and expiration time T. Note: S is fixed in all of the formulas below.

- $C(S, t; K, T) = C(S, 0; K, T - t)$
- $C(aS, t; aK, T) = aC(S, t; K, T)$ for all $a > 0$
- $C(S, t; K, T)$ is an increasing function of S
- $C(S, t; K, T)$ is a decreasing function of $0 \le t \le T$
- $C(S, t; K, T) \to (S - K)_+$ as $t \to T$
- $0 \le \frac{\partial}{\partial S} C(S, t; K, T) \le 1$
- If $S > K$ then $\frac{\partial}{\partial S} C(S, t; K, T) \to 1$ as $t \to T$
- If $S < K$ then $\frac{\partial}{\partial S} C(S, t; K, T) \to 0$ as $t \to T$

Envoi

I hope that at least some of these problems are interesting to you. Perhaps one or two may even offer a mild epiphany. They have been created for your enjoyment.

These problems should be fully accessible to everyone. Still, easy or hard, I hope that at least a few will scratch out some higher, more conceptual messages. That is the intention behind their design.

There are a few problems that will provide some challenge to almost anyone, but even if you took a brief "break" from the course, don't count yourself out. You can still do all of these problems if that is your desire and if you have time to give them an honestly try.

Still, life is short. You should do the problems you want to do and skip the rest. Whatever problems you chose to solve, I promise to read carefully what you have written carefully. I will do my best to understand your ideas.

Good luck to all!