A SIMPLE PROOF OF A RESULT OF A. NOVIKOV

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Abstract. We give simple proofs that for a continuous local martingale M_t

$$\liminf_{\varepsilon\downarrow 0}\varepsilon\log Ee^{(1-\varepsilon)\langle M\rangle_\infty/2}<\infty\Longrightarrow E\exp(M_\infty-\langle M\rangle_\infty/2)=1,$$

$$\liminf_{\varepsilon\downarrow 0}\varepsilon\log\sup_{t\geq 0}Ee^{(1-\varepsilon)M_t/2}<\infty\Longrightarrow E\exp(M_\infty-\langle M\rangle_\infty/2)=1.$$

1. Let $(\Omega,\mathcal{F},P)$ be a complete probability space and let M_t be a continuous local martingale on $(\Omega,\mathcal{F},P)$ such that $\langle M\rangle=\langle M\rangle_\infty<\infty$ (a.s.). Define

$$M=M_\infty,\ \ \rho=\rho(M)=e^{M-\langle M\rangle/2},\ \ \rho_t=\rho_t(M)=e^{M_t-\langle M\rangle_t/2}.$$

We will be discussing generalizations of the following result of A. Novikov (1973):

$$Ee^{\langle M\rangle/2}<\infty\Longrightarrow E\rho=1.$$

This result is quite important in many applications related to absolute continuous change of probability measure and, in particular, makes available Girsanov's theorem.

The original proof and other known proofs are rather complicated, and here we want first to present an elementary proof of a somewhat stronger result

$$\lim_{\varepsilon\downarrow 0}\varepsilon\log Ee^{(1-\varepsilon)\langle M\rangle/2}=0\Longrightarrow E\rho=1. \tag{1}$$

Then in n. 4 we show that $=0$ in (1) can be replaced with $<\infty$.

It turns out that to prove (1) it only suffices to use the following two facts:

$$E\rho\leq 1;\ \ \exists\varepsilon>0: Ee^{(1+\varepsilon)\langle M\rangle/2}<\infty\Longrightarrow E\rho=1. \tag{2}$$

Indeed, if we accept (2), then under the condition in (1)

$$Ee^{(1+\varepsilon)^2\langle(1-\varepsilon)M\rangle/2}=Ee^{(1-\varepsilon^2)^2\langle M\rangle/2}<\infty,$$

which by (2) and by Hölder's inequality implies that

$$1=E\rho((1-\varepsilon)M)=Ee^{(1-\varepsilon)(M-\langle M\rangle/2)}e^{(1-\varepsilon)\varepsilon\langle M\rangle/2}$$
$$\leq(E\rho)^{1-\varepsilon}(Ee^{(1-\varepsilon)\langle M\rangle/2})^{\varepsilon}.$$

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By letting $\varepsilon \downarrow 0$ we get $1 \le E\rho$, which together with the first relation in (2) implies our statement (1).

2. In the same way we can improve a result of N. Kazamaki (1978). Namely we claim that

$$\lim_{\varepsilon\downarrow 0} \varepsilon \log \sup_{t\ge 0} Ee^{(1-\varepsilon)M_t/2} = 0 \Longrightarrow E\rho = 1. \tag{3}$$

Here we use that

$$\exists \varepsilon > 0 : \sup_{t\ge 0} Ee^{(1+\varepsilon)M_t/2} < \infty \Longrightarrow E\rho = 1. \tag{4}$$

Then under the condition in (3) we have

$$\sup_{t\ge 0} Ee^{(1+\varepsilon)\{(1-\varepsilon)M_t\}/2} = \sup_{t\ge 0} Ee^{(1-\varepsilon^2)M_t/2} < \infty,$$

$$\begin{aligned} 1 = E\rho((1-\varepsilon)M) &= Ee^{(1-\varepsilon)^2(M-\langle M\rangle/2)}e^{(1-\varepsilon)\varepsilon M} \\ &\le (E\rho)^{(1-\varepsilon)^2}(Ee^{(1-\varepsilon/(2-\varepsilon))M/2})^{\varepsilon(2-\varepsilon)}, \end{aligned}$$

and we conclude as before.

3. Assertions (1) and (3) are stronger than the corresponding results of A. Novikov and N. Kazamaki. To show this for (1), take a one-dimensional Wiener process w_t and let τ be the first exit time of w_t from $(-\pi, \pi)$. Define $M_t = w_{t\wedge\tau}/4$. Then one can easily see that for $\varepsilon \downarrow 0$

$$[Ee^{(1-\varepsilon)\langle M\rangle/2)}]^\varepsilon = [Ee^{(1-\varepsilon)\tau/8}]^\varepsilon = \big[\cos\frac{\sqrt{1-\varepsilon}}{2}\pi\big]^{-\varepsilon} \to 1,$$

so that the assumption in (1) is satisfied, whereas $E\exp(\langle M\rangle/2) = \infty$ and Novikov's criterion is not applicable.

In the case of (3) take τ to be an exponentially distributed random variable independent of w. Specifically, let $P(\tau > t) = e^{-t/2}$ and define $M = 2w_{t\wedge\tau}$. Then

$$\begin{aligned} &[\sup_{t\ge 0} Ee^{(1-\varepsilon)M_t/2}]^\varepsilon \\ &\quad = \left[\sup_{t\ge 0}\frac{1}{2}\left\{\int_0^t e^{-s/2}Ee^{(1-\varepsilon)w_s}\,ds + Ee^{(1-\varepsilon)w_t}\int_t^\infty e^{-s/2}\,ds\right\}\right]^\varepsilon \\ &\quad = \left[\frac{1}{2}\int_0^\infty e^{-s(1-(1-\varepsilon)^2)/2}\,ds\right]^\varepsilon \to 1. \end{aligned}$$

At the same time $\sup_{t\ge 0} E\exp(M_t/2) = \infty$, and Kazamaki's criterion is not applicable.

4. Now we show further improvements of (1) and (3):

$$\varliminf_{\varepsilon\downarrow 0} \varepsilon \log Ee^{(1-\varepsilon)\langle M\rangle/2} < \infty \Longrightarrow E\rho = 1, \tag{5}$$

$$\varliminf_{\varepsilon\downarrow 0} \varepsilon \log \sup_{t\ge 0} Ee^{(1-\varepsilon)M_t/2} < \infty \Longrightarrow E\rho = 1. \tag{6}$$

To prove (5) we proceed as in the proof of (1) and we write

$$1 = E\rho((1-\varepsilon)M) = Ee^{(1-\varepsilon)(M-\langle M\rangle/2)}e^{(1-\varepsilon)\varepsilon\langle M\rangle/2}I_{\langle M\rangle\le T}$$
$$+EI_{\langle M\rangle>T}e^{(1-\varepsilon)(M-\langle M\rangle/2)}e^{(1-\varepsilon)\varepsilon\langle M\rangle/2}$$
$$\le (E\rho)^{1-\varepsilon}(Ee^{(1-\varepsilon)\langle M\rangle/2}I_{\langle M\rangle\le T})^{\varepsilon} + (E\rho I_{\langle M\rangle>T})^{1-\varepsilon}(Ee^{(1-\varepsilon)\langle M\rangle/2})^{\varepsilon},$$

where T is a constant, $T\in(0,\infty)$. As $\varepsilon\downarrow 0$, we get

$$1 \le E\rho + \text{const}\, E\rho I_{\langle M\rangle>T},$$

which gives $1\le E\rho$ after letting $T\to\infty$. In like manner (6) is proved.

5. For the sake of completeness we also present the proofs of (2) and (4). The first relation in (2) is true because ρ_t is a nonnegative local martingale (by Itô's formula). From this we get

$$Ee^{M_t/2} = Ee^{M_t/2-\langle M\rangle_t/4}e^{\langle M\rangle_t/4} \le (Ee^{\langle M\rangle_t/2})^{1/2} \le (Ee^{\langle M\rangle/2})^{1/2},$$

and it remains only to prove (4). To do this step take $\delta>0$ and $p>1$, define $\gamma=(p(1+\delta))^{-1/2}$ and $q=p(p-1)^{-1}$, and observe that by Doob's inequality for moments of the local martingale ρ_t

$$E\sup_{t>0}\rho_t^{1+\delta} \le N\sup_{t>0}E\rho_t^{1+\delta} = N\sup_{t>0}Ee^{\gamma(1+\delta)M_t-(1+\delta)\langle M\rangle_t/2}e^{(1-\gamma)(1+\delta)M_t}$$
$$\le N\sup_{t>0}(E\rho_t(p\gamma(1+\delta)M))^{1/p}(Ee^{\kappa M_t})^{1/q} \le N\sup_{t>0}(Ee^{\kappa M_t})^{1/q},$$

where $\kappa=(1-\gamma)(1+\delta)q$ and N is a constant. If $\delta=0$ and $p\downarrow 1$, then $\kappa\downarrow 1/2$. Therefore, given that the condition in (4) is satisfied, we can find $\delta>0$ and $p>1$ such that $\kappa\le(1+\varepsilon)/2$, and then $E\sup_{t>0}\rho_t^{1+\delta}<\infty$ and $E\sup_{t>0}\rho_t<\infty$. Finally we take a sequence of stopping times τ_n localizing ρ and, by using the dominated convergence theorem, conclude

$$E\rho = E\lim_{n\to\infty}\rho_{\tau_n} = \lim_{n\to\infty}E\rho_{\tau_n} = 1.$$

References

[1] Kazamaki N. (1978) *The equivalence of two conditions on weighted norm inequalities for martingales*, in: Proc. Intern. Symp. SDE Kyoto 1976, K. Itô (Ed.), Kinokuniya, Tokyo, pp. 141-152.

[2] Novikov A.A. (1973) *On moment inequalities and identities for stochastic integrals*, Proc. Second Japan–USSR Symp. Prob. Theory, Lecture Notes in Math., Vol. 330, 333-339.

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