POSITIVE DEFINITE MATRIX PROBLEM — SOLUTIONS

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Abstract. The self-imposed rule of the *Cauchy-Schwarz Master Class* was to keep matrix algebra to a bare minimum. This decision was made to impose a discipline of simplicity, but many babies were thrown out with the bath water. Here is one that is certainly simple enough to have been include, even as a warm-up problem. It's also useful.

Problem: *Give a necessary and sufficient condition on α and β in order that*

$$T^2 + \alpha T + \beta I$$

be positive definite for each self-adjoint matrix T.

Solution: To find a necessary condition, take T to be the 1-dimensional identity matrix that takes 1 to x. We then need

$$0 < \langle T^2 + \alpha T + \beta I, T^2 + \alpha x + \beta I \rangle = x^2 + \alpha x + \beta 1.$$

By the solution of the quadratic equation, for this to be positive for all x requires that

$$\alpha^2 < 4\beta. \tag{1}$$

This gives us a nice necessary condition which will be sufficient if we are lucky.

Now take any self-adjoint T, take any v with $\langle v, v \rangle = 1$, and write out the natural bounds:

$$\begin{aligned}\langle T^2 + \alpha T + \beta I, T^2 + \alpha + \beta I v, v \rangle &= \langle T^2 v, v \rangle + \alpha \langle T v, v \rangle + \beta \langle v, v \rangle \\ &= \langle T v, T v \rangle + \alpha \langle T v, v \rangle + \beta \langle v, v \rangle \\ &\geq ||Tv||^2 - |\alpha| ||Tv|| + \beta\end{aligned}$$

where in the last line we let Cauchy-Schwarz have its day.

We can now say that the quadratic formula implies that by our hypothesis (1), but its more fun and more explicit simply rewrite our bounding term as

$$(||Tv|| - |\alpha|/2)^2 + (\beta - \alpha^2/4),$$

which certainly does the trick. The first term is a square and the second is strictly positive precisely under our hypothesis (1).

More Simple—but Nice—Norm Facts

Problem. (a) Show that $\langle u, v\rangle = 0$ if and only if

$$||u|| \leq ||u + \alpha v|| \quad \text{for all } \alpha \in \mathbb{R}.$$

(b) Given orthonormal vectors $e_1, e_2, ..., e_n$, show that v is in their linear span if and only if

$$\langle v, v\rangle = |\langle v, e_1\rangle|^2 + |\langle v, e_2\rangle|^2 + \cdots + |\langle v, e_n\rangle|^2.$$

Remarks. These exercises are based on Axler (2000, pp.122–123), and they are interesting since they provide *characterizations*. They would be most instructive if served up as part of a suite of problems on limit results which exploit characterization. Incidentally, this kind of property has been used to define "orthogonality" in normed spaces that are not inner product spaces.

References

[1] Axler, S., *Linear Algebra Done Right*, 2nd ed., Springer, New York, 2000.

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