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Optimal online selection of a monotone subsequence: a central limit theorem

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Abstract

Consider a sequence of n independent random variables with a common continuous distribution F, and consider the task of choosing an increasing subsequence where the observations are revealed sequentially and where an observation must be accepted or rejected when it is first revealed. There is a unique selection policy π_n^* that is optimal in the sense that it maximizes the expected value of $L_n(\pi_n^*)$, the number of selected observations. We investigate the distribution of $L_n(\pi_n^*)$; in particular, we obtain a central limit theorem for $L_n(\pi_n^*)$ and a detailed understanding of its mean and variance for large n. Our results and methods are complementary to the work of Bruss and Delbaen (2004) where an analogous central limit theorem is found for monotone increasing selections from a finite sequence with cardinality N where N is a Poisson random variable that is independent of the sequence.

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1. Introduction

In the problem of *online* selection of a *monotone increasing* subsequence, a decision maker observes a sequence of independent non-negative random variables $\{X_1, X_2, \ldots, X_n\}$ with common continuous distribution F, and the task is to select a subsequence $\{X_{\tau_1}, X_{\tau_2}, \ldots, X_{\tau_j}\}$ such that

$$X_{\tau_1} \le X_{\tau_2} \le \cdots \le X_{\tau_j}$$

where the indices $1 \le \tau_1 < \tau_2 < \cdots < \tau_j \le n$ are stopping times with respect to the σ-fields $\mathcal{F}_i = \sigma\{X_1, X_2, \ldots, X_i\}$, $1 \le i \le n$. In other words, at time i when the random variable X_i is first observed, the decision maker has to choose to accept X_i as a member of the monotone increasing sequence that is under construction, or to reject X_i from any further consideration.

We call such a sequence of stopping times a *feasible policy*, and we denote the set of all such policies by $\Pi(n)$. For any $\pi \in \Pi(n)$, we then let $L_n(\pi)$ be the random variable that counts the number of selections made by policy π for the realization $\{X_1, X_2, \ldots, X_n\}$; that is,

$$L_n(\pi) = \max\{j : X_{\tau_1} \le X_{\tau_2} \le \cdots \le X_{\tau_j} \text{ where } 1 \le \tau_1 < \tau_2 < \cdots < \tau_j \le n\}.$$

Samuels and Steele [25] found that for each $n \ge 1$ there is a unique policy $\pi_n^* \in \Pi(n)$ such that

$$\mathbb{E}[L_n(\pi_n^*)] = \sup_{\pi \in \Pi(n)} \mathbb{E}[L_n(\pi)], \tag{1}$$

and for such optimal policies one has

$$\mathbb{E}[L_n(\pi_n^*)] \sim (2n)^{1/2} \quad \text{as } n \to \infty. \tag{2}$$

Bruss and Robertson [10] and Gnedin [12] showed that one actually has the crisp upper bound

$$\mathbb{E}[L_n(\pi_n^*)] \le (2n)^{1/2} \quad \text{for all } n \ge 1, \tag{3}$$

and, as corollaries of related work, Rhee and Talagrand [23], Gnedin [12] and Arlotto and Steele [3] all found that there is an asymptotic error rate for the lower bound

$$(2n)^{1/2} - O(n^{1/4}) \le \mathbb{E}[L_n(\pi_n^*)] \quad \text{as } n \to \infty. \tag{4}$$

Here, our main goal is to show that $L_n(\pi_n^*)$ satisfies a central limit theorem.

Theorem 1 (*Central Limit Theorem for Optimal Online Monotone Selections*)**.** *For any continuous distribution F one has for $n \to \infty$ that*

$$(2n)^{1/2} - O(\log n) \le \mathbb{E}[L_n(\pi_n^*)] \le (2n)^{1/2}, \tag{5}$$

$$\frac{1}{3}\mathbb{E}[L_n(\pi_n^*)] - O(1) \le \mathrm{Var}[L_n(\pi_n^*)] \le \frac{1}{3}\mathbb{E}[L_n(\pi_n^*)] + O(\log n), \tag{6}$$

and one has the convergence in distribution

$$\frac{3^{1/2}\{L_n(\pi_n^*) - (2n)^{1/2}\}}{(2n)^{1/4}} \Longrightarrow N(0, 1). \tag{7}$$

Two connections help to put this result in context. First, it is useful to recall the analogous problem of offline (or full information) subsequence selection, for which there is a remarkably

rich literature. Second, there are closely related results of Bruss and Delbaen [8,9] that deal with sequential selection where the number of values to be seen is random with a Poisson distribution.

First connection: the Tracy–Widom law

If one knows all of the values $\{X_1, X_2, \ldots, X_n\}$ at the time the selections begin, then decision maker can select a maximal increasing subsequence with length

$$L_n = \max\{j : X_{i_1} \le X_{i_2} \le \cdots \le X_{i_j} \text{ where } 1 \le i_1 < i_2 < \cdots < i_j \le n\}. \tag{8}$$

This *full information* or *offline* length L_n has been studied extensively.

The question of determining the distribution of L_n was first raised by Ulam [27], but the analysis of L_n was taken up in earnest by Hammersley [14], Kingman [17], Logan and Shepp [19], and Veršik and Kerov [28] who established in steps that

$$\mathbb{E}[L_n] \sim 2n^{1/2} \quad \text{as } n \to \infty.$$

Much later H. Kesten conjectured (cf. Aldous and Diaconis [1, p. 416]) that there should be positive constants α and β such that

$$\mathbb{E}[L_n] = 2n^{1/2} - \alpha n^{1/6} + o(n^{1/6}) \quad \text{and} \quad \{\mathrm{Var}[L_n]\}^{1/2} = \beta n^{1/6} + o(n^{1/6}). \tag{9}$$

After subtle progress by Pilpel [22], Bollobás and Brightwell [5], Kim [16], Bollobás and Janson [6], and Odlyzko and Rains [21] this conjecture was settled affirmatively by Baik, Deift and Johansson [4] who proved moreover that $n^{-1/6}(L_n - 2n^{1/2})$ converges in distribution to the famous Tracy–Widom law which had emerged just a bit earlier from the theory of random matrices. The recent monograph of Romik [24] gives a highly readable account of this development.

One distinction between the online and the offline problems is that, while the means are of the same order in each case, the variances are not of the same order. The standard deviation for offline selection is of order $n^{1/6}$, but by (6) the standard deviation for the online selection is of order $n^{1/4}$. Intuitively this difference reflects greater uncertainty in the online selection problem than in the offline problem, but it is harder to imagine why moving to the online formulation would drive one all of the way from the Tracy–Widom law to the Gaussian law.

Second connection: the Bruss–Delbaen central limit theorem

Consider the problem of sequential selection of a monotone increasing subsequence from $\{X_1, X_2, \ldots, X_{N_\nu}\}$ where N_ν is a Poisson random variable with mean ν that is independent of the sequence $\{X_1, X_2, \ldots\}$. Just as in (1) there is a unique sequential policy that maximizes the expected number of selections that are made. If we denote this optimal policy by $\pi^*_{N_\nu}$ then as before $L_{N_\nu}(\pi^*_{N_\nu})$ is the number of selections from $\{X_1, X_2, \ldots, X_{N_\nu}\}$ that are made by the policy $\pi^*_{N_\nu}$.

Bruss and Delbaen [8] proved that, as $\nu \to \infty$, one has the mean estimate

$$\mathbb{E}[L_{N_\nu}(\pi^*_{N_\nu})] = (2\nu)^{1/2} + O(\log \nu), \tag{10}$$

and the variance estimate

$$\mathrm{Var}[L_{N_\nu}(\pi^*_{N_\nu})] = \frac{1}{3}(2\nu)^{1/2} + O(\log \nu).$$

Moreover, Bruss and Delbaen [9] proved that, as $\nu \to \infty$, one has the convergence in distribution

$$\frac{3^{1/2}\{L_{N_\nu}(\pi^*_{N_\nu}) - (2\nu)^{1/2}\}}{(2\nu)^{1/4}} \Longrightarrow N(0, 1).$$

One needs to ask if it is possible to "de-Poissonize" these results to get Theorem 1, either in whole or in part. We show in Section 3 that the lower half of (5) can be obtained from (10) by an easy de-Poissonization argument; in fact, this is the only proof we know of this bound. In Section 3 we also explain as best we can, why no further parts of Theorem 1 can be obtained by de-Poissonization.

One can further ask if it might be possible to adapt the *methods* of Bruss and Delbaen [8,9] to prove Theorem 1. The major benefit of a Poisson horizon is that it gives access to the tools of continuous time Markov processes such as the infinitesimal generator and Dynkin's martingales. Moreover, in this instance the associated value function $V(t, x)$ can be written as a function of one variable by the space–time transformation $V(t, x) = \bar{V}(t(1 - x))$.

Here we lack these benefits. We work in discrete time with a known finite horizon, and our value function $v_k(s)$ permanently depends on the state s and the time to the horizon k. This puts one a long way from the world of Bruss and Delbaen [8,9]. Still, in Section 7 we give a brief proof of the well-known upper bound (3) that echoes an argument of Bruss and Delbaen [9, pp. 291–292]. This seems to be the only instance of an overlap of technique.

Organization of the analysis

The proof of our central limit theorem has two phases. In the first phase, we investigate the analytic properties of the value functions given by framing the selection problem as a Markov decision problem. Section 2 addresses the monotonicity and the submodularity of the value functions. We also obtain that the map $n \mapsto \mathbb{E}[L_n(\pi_n^*)]$ is concave, and this is used in Section 3 to prove the lower half of (5); this is our only de-Poissonization argument.

Sections 4 and 5 develop smoothness and curvature properties of the value functions. In particular, we find that in the uniform model the value functions are concave as a function of the state variable, but, for the exponential model, they are convex. This broken symmetry is surprisingly useful even though the distribution of $L_n(\pi_n^*)$ does not depend on the model distribution F.

The second phase of the proof deals with a natural martingale that one obtains from the value functions. This martingale is defined in Section 6, and it is used in Sections 7–9 to estimate the conditional variances of $L_n(\pi_n^*)$. These estimates and a martingale central limit theorem are then used in Section 10 to complete the proof of Theorem 1. Finally, in Section 11 we comment briefly on two open problems and the general nature of the methods developed here.

2. Structure of the value functions

We now let $v_k(s)$ denote the expected value of the number of monotone increasing selections under the optimal policy when (i) there are k observations that remain to be seen and (ii) the value of the most recently selected observation is equal to s. The functions $\{v_k : 1 \le k < \infty\}$ are called the *value functions*, and they can be determined recursively. Specifically, we have the *terminal condition*

$$v_0(s) = 0 \quad \text{for all } s \ge 0,$$

and if we set $F(s) = P(X_i \le s)$ then for all $k \ge 1$ and $s \ge 0$ we have the recursion

$$v_k(s) = F(s)v_{k-1}(s) + \int_s^\infty \max\{v_{k-1}(s), 1 + v_{k-1}(x)\}\, dF(x). \tag{11}$$

To see why this equation holds, note that with probability $F(s)$ one is presented at time $i = n - k + 1$ with a value X_i that is less than the previously selected value s. In this situation, we

do not have the opportunity to select X_i. This leaves us with $k-1$ observations to be seen and with the value of the last selected observation, s, unchanged. This possibility contributes the term $F(s)v_{k-1}(s)$ to our equation.

Now, if the newly presented value satisfies $s \le X_i$ then we have the *option* to select or reject $X_i = x$. If we select $X_i = x$, then the sum of our present reward and expected future reward is $1 + v_{k-1}(x)$. On the other hand, if we choose not to select $X_i = x$, then we have no present reward and the expected future reward is $v_{k-1}(s)$ since the value of the running maximum is not changed. Since X_i has distribution F, the expected optimal contribution is given by the second term of Eq. (11).

The identity (11) is called the *Bellman recursion* for the sequential selection problem. In principle, it tells us everything there is to know about the value functions; in particular, it determines

$$\mathbb{E}[L_n(\pi_n^*)] = v_n(0) \quad \text{for all } n \ge 1.$$

Qualitative information can also be extracted from the recursion (11). For example, it is immediate from (11) that the value functions are always continuous. More refined properties of the value functions may depend on F, and here it is often useful to consider a special subclass of distributions.

Definition 2 (*Admissible Distribution*)**.** A distribution F is said to be *admissible* if there is an open interval $\mathcal{I} \subseteq [0, \infty)$ such that

(i) F is differentiable on $\mathcal{I}$,
(ii) $F'(x) = f(x) > 0$ for all $x \in \mathcal{I}$, and
(iii) $\int_{\mathcal{I}} f(x)\,dx = 1$.

The next lemma illustrates how admissibility can be used. The result is largely intuitive, but the formal proof via (11) suggests that some care is needed.

Lemma 3 (*Monotonicity of Value Functions*)**.** *For any distribution F the value functions are non-increasing. Moreover, if F is admissible, then the value functions are strictly decreasing on $\mathcal{I}$.*

Proof. The first assertion is trivial, so we focus on the second. To organize our induction we denote by $\mathcal{H}_k$ the assertion

$$v_k(s+\epsilon) < v_k(s) \quad \text{for all } s \in \mathcal{I} \text{ and all } \epsilon > 0.$$

When $k = 1$, we have $v_1(s) = 1 - F(s)$, and admissibility of F implies v_1 is strictly decreasing on $\mathcal{I}$. This establishes the base case $\mathcal{H}_1$.

For $k > 1$ we assume that $\mathcal{H}_{k-1}$ holds, and we note by the Bellman recursion (11) and the characterizing properties of admissible distributions that

$$\begin{aligned}
v_k(s+\epsilon) - v_k(s) &= F(s+\epsilon)v_{k-1}(s+\epsilon) \\
&\quad + \int_{s+\epsilon}^{\infty} \max\{v_{k-1}(s+\epsilon), 1 + v_{k-1}(x)\} f(x)\,dx \\
&\quad - F(s)v_{k-1}(s) - \int_{s}^{\infty} \max\{v_{k-1}(s), 1 + v_{k-1}(x)\} f(x)\,dx \\
&\le F(s+\epsilon)v_{k-1}(s+\epsilon) + \int_{s+\epsilon}^{\infty} \max\{v_{k-1}(s), 1 + v_{k-1}(x)\} f(x)\,dx \\
&\quad - F(s+\epsilon)v_{k-1}(s) - \int_{s+\epsilon}^{\infty} \max\{v_{k-1}(s), 1 + v_{k-1}(x)\} f(x)\,dx \\
&= F(s+\epsilon)\left\{v_{k-1}(s+\epsilon) - v_{k-1}(s)\right\},
\end{aligned}$$

where we first used $v_{k-1}(s+\epsilon) < v_{k-1}(s)$ and then used the trivial estimate

$$\{F(s+\epsilon) - F(s)\}v_{k-1}(s) \le \int_s^{s+\epsilon} \max\{v_{k-1}(s), 1 + v_{k-1}(x)\} f(x)\, dx.$$

For $s \in \mathcal{I}$ one has strict positivity of $F(s+\epsilon)$, so by the induction hypothesis $\mathcal{H}_{k-1}$ we have $v_k(s+\epsilon) - v_k(s) \le F(s+\epsilon)\{v_{k-1}(s+\epsilon) - v_{k-1}(s)\} < 0$. □

Optimal threshold functions

The monotonicity of v_{k-1} tells us that the integrand in (11) equals the right maximand $\{1 + v_{k-1}(x)\}$ on a certain initial segment of $[s, \infty)$, and it equals the left maximand $v_{k-1}(s)$ on the rest of the segment. This observation leads to a useful reformulation of the Bellman recursion; specifically, if we set

$$h_k(s) = \sup\{x \in [s, \infty) : F(x) < 1 \text{ and } v_{k-1}(s) \le 1 + v_{k-1}(x)\}, \tag{12}$$

then the Bellman recursion (11) can be written as

$$v_k(s) = \{1 - F(h_k(s)) + F(s)\}v_{k-1}(s) + \int_s^{h_k(s)} \{1 + v_{k-1}(x)\}\, dF(x). \tag{13}$$

The functions $\{h_k : 1 \le k < \infty\}$ defined by (12) are called the *optimal threshold functions*.

If $v_{k-1}(s) \le 1$, the characterization (12) has an informative policy interpretation. Namely, if $v_{k-1}(s) \le 1$, then the optimal strategy for the decision maker is the greedy strategy where one accepts any arriving observation that is as large as s. On the other hand, if $v_{k-1}(s) > 1$, the optimal decision maker needs to act more conservatively; when k observations remain to be seen, one only accepts the newly arriving observation if it falls in the interval $[s, h_k(s)]$.

When F is admissible, we have the strict monotonicity of v_{k-1}, and this allows a second characterization of the threshold function:

$$h_k(s) \text{ uniquely satisfies } v_{k-1}(s) = 1 + v_{k-1}(h_k(s)) \quad \text{if } v_{k-1}(s) > 1. \tag{14}$$

The value $h_k(s)$ of the threshold function thus marks the point of indifference between the optimal acceptance region and the optimal rejection region. The characterization (14) also motivates a definition.

Definition 4 (*Critical Value*)**.** If F is admissible, then the unique solution of the equation $v_k(s) = 1$ is called the *critical value*, and it is denoted by s_k^*.

The analytical character of h_k changes at s_k^*, and one has to be attentive to the differing behavior of h_k above and below s_k^*. We will not need this distinction until Section 4, but it is critical there.

We complete this section by recording two simple (but useful) bounds on the time-difference of the value function. These bounds follow from the characterization (12) for the optimal threshold h_k and the monotonicity of the value function v_{k-1}.

Lemma 5. *For $s \ge 0$ and $1 \le k < \infty$, we have the inequalities*

$$0 \le v_k(s) - v_{k-1}(s) \le F(h_k(s)) - F(s) \le 1. \tag{15}$$

From a modeler's perspective, this inequality is intuitive since $F(h_k(s)) - F(s)$ can be interpreted as the probability that one selects the next observation when k observations remain

to be seen. A formal confirmation of (15) illustrates the handiness of the second form (13) of the Bellman equation.

Proof of Lemma 5. First, note that after subtracting $v_{k-1}(s)$ from both sides of Eq. (13), we have

$$v_k(s) - v_{k-1}(s) = \int_s^{h_k(s)} \{1 + v_{k-1}(x) - v_{k-1}(s)\}\, dF(x).$$

The map $x \mapsto v_{k-1}(x)$ is monotone decreasing, so the factor $\{1 + v_{k-1}(x) - v_{k-1}(s)\}$ is bounded above by one. This gives us our upper bound in (15). The representation (12) for h_k tells us the integrand is non-negative on $[s, h_k(s)]$, and this gives the lower bound in (15). □

Value function submodularity

If one increases the number k of observations yet to be seen, then the decision maker faces a richer set of future possibilities. This in turn suggests that the decision maker may want to act more conservatively, keeping more powder dry for future action. Specifically, one might guess that $h_{k+1}(s) \le h_k(s)$ for all $s \in [0, \infty)$ and all $1 \le k < \infty$. We confirm this guess as a corollary of the next proposition which gives us a pivotally useful property of the value functions.

Proposition 6 (*Submodularity of the Value Functions*)**.** *The sequence of value functions* $\{v_k : 1 \le k < \infty\}$ *determined by the Bellman recursion* (11) *is* submodular *in the sense that for all* $1 \le k < \infty$ *one has*

$$v_{k-1}(s) - v_{k-1}(t) \le v_k(s) - v_k(t) \quad \textit{for all } 0 \le s \le t < \infty. \tag{16}$$

Proof. We first derive a recursion for the difference $v_k(s) - v_k(t)$. For $0 \le s \le t < \infty$, we have from (11) that

$$\begin{aligned} v_k(s) - v_k(t) \;=\; & F(s)\{v_{k-1}(s) - v_{k-1}(t)\} \\ & + \int_s^t \max\{v_{k-1}(s) - v_{k-1}(t), 1 + v_{k-1}(x) - v_{k-1}(t)\}\, dF(x) \\ & + \int_t^\infty [\max\{v_{k-1}(s), 1 + v_{k-1}(x)\} \\ & - \max\{v_{k-1}(t), 1 + v_{k-1}(x)\}]\, dF(x). \end{aligned} \tag{17}$$

Next, we let

$$a_{k-1}(x) \stackrel{\text{def}}{=} \min\{v_{k-1}(s) - v_{k-1}(t), v_{k-1}(s) - v_{k-1}(x) - 1\},$$
$$b_{k-1}(x) \stackrel{\text{def}}{=} \min\{1 + v_{k-1}(x) - v_{k-1}(t), 0\},$$

and we note that the difference

$$\max\{v_{k-1}(s), 1 + v_{k-1}(x)\} - \max\{v_{k-1}(t), 1 + v_{k-1}(x)\}$$

which appears in the last integrand of (17) can be written as

$$\max\{v_{k-1}(s), 1 + v_{k-1}(x)\} - \max\{v_{k-1}(t), 1 + v_{k-1}(x)\} = \max\{a_{k-1}(x), b_{k-1}(x)\}.$$

Here $s \le t$, so when $b_{k-1}(x) < 0$ the monotonicity of the value functions in Lemma 3 implies that $0 \le v_{k-1}(s) - v_{k-1}(x) - 1$. It then follows that $0 \le a_{k-1}(x)$ and $\max\{a_{k-1}(x), b_{k-1}(x)\} =$

$\max\{a_{k-1}(x), 0\}$. In general, we then have the equivalence

$$\max\{v_{k-1}(s), 1 + v_{k-1}(x)\} - \max\{v_{k-1}(t), 1 + v_{k-1}(x)\} = \max\{a_{k-1}(x), 0\},$$

and we can substitute this representation and the explicit expression for $a_{k-1}(x)$ in (17) to obtain the simplified *difference recursion*

$$\begin{aligned} v_k(s) - v_k(t) &= F(s)\{v_{k-1}(s) - v_{k-1}(t)\} \\ &\quad + \int_s^t \max\{v_{k-1}(s) - v_{k-1}(t), 1 + v_{k-1}(x) - v_{k-1}(t)\}\, dF(x) \\ &\quad + \int_t^\infty \max\{\min\{v_{k-1}(s) - v_{k-1}(t), v_{k-1}(s) - v_{k-1}(x) - 1\}, 0\}\, dF(x). \end{aligned} \tag{18}$$

We now let $\mathcal{H}_k$ be the assertion that

$$v_{k-1}(s) - v_{k-1}(t) \le v_k(s) - v_k(t) \quad \text{for all } 0 \le s \le t < \infty,$$

and we prove by induction that $\mathcal{H}_k$ holds for all $k \ge 1$. We first note that for $k = 1$ we have $v_0(s) = 0$ for all $s \in [0, \infty)$. By the difference recursion (18) we obtain

$$v_1(s) - v_1(t) = F(t) - F(s) \ge 0 = v_0(s) - v_0(t),$$

so the base case $\mathcal{H}_1$ holds.

Next, we suppose that $\mathcal{H}_{k-1}$ holds, and we apply $\mathcal{H}_{k-1}$ to *all* of the terms on the right-hand side of (18). We then obtain that

$$\begin{aligned} v_k(s) - v_k(t) \le F(s)\{v_k(s) - v_k(t)\} &+ \int_s^t \max\{v_k(s) - v_k(t), 1 + v_k(x) - v_k(t)\}\, dF(x) \\ &+ \int_t^\infty \max\{\min\{v_k(s) - v_k(t), v_k(s) - v_k(x) - 1\}, 0\}\, dF(x). \end{aligned}$$

We can now apply the difference recursion (18) a second time after we replace k by $k + 1$. This tells us that the right-hand side above is equal to the difference $v_{k+1}(s) - v_{k+1}(t)$, thus completing the proof of $\mathcal{H}_k$ and of the proposition. □

The submodularity guaranteed by Proposition 6 is more powerful than one might expect. In particular, it delivers three basic corollaries.

Corollary 7 (*Monotonicity of Optimal Thresholds*)**.** *For the threshold functions characterized by (12) we have*

$$h_{k+1}(s) \le h_k(s) \quad \textit{for } 0 \le s < \infty. \tag{19}$$

Proof. Here we only have to note that

$$\begin{aligned} h_{k+1}(s) &= \sup\{x \in [s, \infty) : F(x) < 1 \text{ and } v_k(s) - v_k(x) \le 1\} \\ &\le \sup\{x \in [s, \infty) : F(x) < 1 \text{ and } v_{k-1}(s) - v_{k-1}(x) \le 1\} \\ &= h_k(s), \end{aligned}$$

where the one inequality comes directly from the submodularity (16) and the two equalities come from (12). □

Corollary 8 (*Concavity in k of the Value Functions.*)**.** *The value functions are concave as functions of k; that is, for each $s \in [0, \infty)$ and all $k \geq 1$, one has*

$$v_{k+1}(s) - 2v_k(s) + v_{k-1}(s) \leq 0.$$

Proof. By the monotonicity (19) of the optimal threshold functions, the recursion (13) gives us the difference identity

$$\begin{aligned} v_{k+1}(s) - v_k(s) &= v_k(s) - v_{k-1}(s) \\ &\quad + \int_s^{h_{k+1}(s)} \{v_{k-1}(s) - v_{k-1}(x) - v_k(s) + v_k(x)\}\, dF(x) \\ &\quad + \int_{h_{k+1}(s)}^{h_k(s)} \{v_{k-1}(s) - 1 - v_{k-1}(x)\}\, dF(x), \end{aligned}$$

and it suffices to check that the two integrands on the right-hand side are non-positive. Non-positivity of the first integrand follows from the submodularity (16), and non-positivity of the second integrand follows from the characterization of $h_k(s)$ in (12). □

Corollary 9 (*Concavity in n of the Expected Length*)**.** *For any continuous F, the map $n \mapsto \mathbb{E}[L_n(\pi_n^*)]$ is concave in n.*

This is just a special case of Corollary 8 (where one just takes $s = 0$ and $k = n$), but, as we will see in Section 3, this concavity carries noteworthy force.

Remark 10 (*Further Context: an Offline Open Problem*)**.** It is not known if the corresponding concavity holds for the *offline* monotone subsequence problem. That is, we do not know if the map $n \mapsto \mathbb{E}[L_n]$ is concave where L_n is defined by (8). In this case we do know

$$\mathbb{E}[L_n] = 2n^{1/2} - \alpha n^{1/6} + o(n^{1/6}),$$

so concavity does seem like a highly plausible conjecture.

3. Intermezzo: possibilities for de-Poissonization

If N is an integer valued random variable, then one can consider the problem of sequential selection of a monotone increasing subsequence from the random length sequence $\mathcal{S} = \{X_1, X_2, \ldots, X_N\}$. Here, as usual, the elements of the sequence are independent with a common continuous distribution F, and they are also independent of N. We also assume that the decision maker knows F and the distribution of N, but the decision maker does not know the value of N until the sequence $\mathcal{S}$ has been exhausted. We let $L_N(\pi)$ denote the number of selections that are made when one follows a policy π for sequential selection from $\mathcal{S}$.

Proposition 11 (*Information Lower Bound*)**.** *If $\mathbb{E}[N] = n$ for some $n \in \mathbb{N}$, then*

$$\mathbb{E}[L_N(\pi)] \leq \mathbb{E}[L_n(\pi_n^*)]. \tag{20}$$

Proof. The policy π is determined before the realization of N is known, and, for any given j, the policy π is suboptimal when it is used for sequential selection from the sequence $\{X_1, X_2, \ldots, X_j\}$. Thus, if we condition on $N = j$, we then have

$$E[L_N(\pi) \mid N = j] \leq \mathbb{E}[L_j(\pi_j^*)]. \tag{21}$$

Now, if we take $\phi : [0, \infty) \to [0, \infty)$ to be the piecewise linear extension of the map

$$j \mapsto \mathbb{E}[L_j(\pi_j^*)],$$

then by Corollary 9 we have that ϕ is also concave. Finally, by the suboptimality (21), the definition of ϕ, and Jensen's inequality we obtain

$$\mathbb{E}[L_N(\pi)] \le \sum_{j=0}^{\infty} \mathbb{E}[L_j(\pi_j^*)]\mathbb{P}(N = j) = \mathbb{E}[\phi(N)] \le \phi(\mathbb{E}[N]) = \mathbb{E}[L_n(\pi_n^*)]. \quad \square$$

The next corollary establishes one of the five assertions of Theorem 1. It is an immediate consequence of Proposition 11 and the lower half of the mean bound (10) from Bruss and Delbaen [8].

Corollary 12. *For any continuous F we have as $n \to \infty$ that*

$$(2n)^{1/2} - O(\log n) \le E[L_n(\pi_n^*)]. \tag{22}$$

This is a notable improvement over the bound (4) that had been established by several earlier investigations; it improves a $O(n^{1/4})$ error bound all the way down to $O(\log n)$. For the central limit theorem (7), one could still get along with a lower bound as weak as $(2n)^{1/2} - o(n^{1/4})$.

De-Poissonization and decision problems

We get the bound (22) by a *de-Poissonization argument* in the sense that a "fixed n" fact is extracted from a "Poisson N" fact. Such arguments are common in computer science, combinatorics and analysis; one finds many examples in Jacquet and Szpankowski [15], Flajolet and Sedgewick [11, Subsection VIII.5.3], and Korevaar [18, Chapter 6]. Still, Proposition 11 is our only instance of a de-Poissonization argument, and the proof of the proposition suggests in part why one may be hard-pressed to find more.

Decision problems are unlike the classical examples mentioned above. The Poisson N problem and the fixed n problem have different optimal policies, and this mismatch forestalls the kind of direct analytical connection one has in the classical examples. Conditioning on $N = j$ does engage the problem, but the suboptimality of the mismatched policy leads only to one-sided relations such as (20) and (21).

4. Smoothness of the value and threshold functions

We need to show that the value functions associated with an admissible distribution F are continuously differentiable on $\mathcal{I}$. As a preliminary step, we consider the differentiability of the threshold functions in a region determined by the critical values s_k^* that were defined in Section 2.

Lemma 13 (*Differentiability of the Threshold Functions*)**.** *Take F to be admissible and take $k > 1$. If v_{k-1} is differentiable on $\mathcal{I}$ and $s \in \mathcal{I} \cap [0, s_{k-1}^*)$, then h_k is differentiable at s, and one has*

$$h_k'(s) = \frac{v_{k-1}'(s)}{v_{k-1}'(h_k(s))} \ge 0. \tag{23}$$

Proof. Set $Q(x, y) = -1 + v_{k-1}(x) - v_{k-1}(y)$. If Q_y denotes the partial derivative of the function Q with respect to y, we know by our hypotheses that Q_y exists, and Lemma 3 implies that the partial derivative Q_y is strictly positive. Now, if (x_0, y_0) satisfies $Q(x_0, y_0) = 0$, then by the implicit function theorem there is a neighborhood $\mathcal{N}_0$ of x_0 where one can solve $Q(x, y) = 0$ uniquely for y, and the solution y is a differentiable function of x for all $x \in \mathcal{N}_0$. Moreover, if $x < s^*_{k-1}$ then (14) tells us $Q(x, y) = 0$ if and only if $y = h_k(x)$, so h_k is differentiable as claimed. Given the differentiability of h_k at s, the formula (23) follows directly from $v_{k-1}(x) = 1 + v_{k-1}(h_k(x))$ by differentiation and the chain rule. The non-negativity of $h'_k(s)$ then follows because the value function v_{k-1} is strictly decreasing. □

Proposition 14 (*Continuous Differentiability of the Value Functions*)**.** *If F is admissible, then for each $1 \le k < \infty$ the value function $s \mapsto v_k(s)$ is continuously differentiable on $\mathcal{I}$, and we have*

$$v'_k(s) = -f(s) + \{1 - F(h_k(s)) + F(s)\}\, v'_{k-1}(s) \quad \text{for } s \in \mathcal{I}. \tag{24}$$

Proof. We argue by induction on k, and we first note for $k = 1$ that $v_1(s) = 1 - F(s)$, so $v'_1(s) = -f(s)$ and (24) holds since $v_0(s) \equiv 0$. Next, we assume by induction that v_{k-1} is continuously differentiable on $\mathcal{I}$. If $s < s^*_{k-1}$ then the induction assumption and Lemma 13 imply that h_k is differentiable at s. We then differentiate (13) to find

$$\begin{aligned} v'_k(s) &= -f(s) + \{1 - F(h_k(s)) + F(s)\}\, v'_{k-1}(s) \\ &\quad + f(h_k(s))\{1 - v_{k-1}(s) + v_{k-1}(h_k(s))\}h'_k(s) \\ &= -f(s) + \{1 - F(h_k(s)) + F(s)\}\, v'_{k-1}(s), \end{aligned}$$

where the last step used the characterization (14) of h_k. Alternatively, if $s > s^*_{k-1}$ we have $F(h_k(s)) = 1$ and (13) says simply that

$$v_k(s) = F(s)v_{k-1}(s) + \int_s^\infty \{1 + v_{k-1}(x)\} f(x)\, dx.$$

Differentiation of this integral then gives us (24). Thus, one has that (24) holds on all of $\mathcal{I}_k = \mathcal{I} \setminus \{s^*_{k-1}\}$. Moreover, taking left and right limits in (24) gives us

$$\lim_{s \nearrow s^*_{k-1}} v'_k(s) = -f(s^*_{k-1}) + F(s^*_{k-1})v'_{k-1}(s^*_{k-1}) = \lim_{s \searrow s^*_{k-1}} v'_k(s).$$

It is almost obvious that these relations imply the continuous differentiability v_k, but to make it crystal clear let γ be the common value of the limits above and define a continuous function $\bar{v} : \mathcal{I} \to \mathbb{R}$ by setting

$$\bar{v}(s) = \begin{cases} v'_k(s) & \text{if } s < s^*_{k-1} \\ \gamma & \text{if } s = s^*_{k-1} \\ v'_k(s) & \text{if } s > s^*_{k-1}. \end{cases}$$

Next, we obtain by piecewise integration that

$$v_k(s) = v_k(0) + \int_0^s \bar{v}(u)\, du \quad \text{for all } s \in \mathcal{I},$$

implying, as expected, that v_k is continuously differentiable on $\mathcal{I}$. □

5. Spending symmetry: curvature of the value functions

For any continuous F the distribution of $L_n(\pi_n^*)$ is the same; this is an invariance property—or a *symmetry*. When one chooses a particular F, say the uniform distribution, there is a sense in which one *spends symmetry*.

All earlier analyses of $L_n(\pi_n^*)$ passed directly to the uniform distribution without any apparent thought about what might be lost or gained by the transition. Still, it does make a difference how one spends this symmetry. The distribution of $L_n(\pi_n^*)$ is insensitive to F, but the value functions are not.

Specifically, for the uniform distribution the value functions are concave, but for the exponential distribution the value functions are convex. This change of curvature gives one access to different estimates. Over the next several sections we see how specialization of the driving distribution has a big influence on the estimation of variances and conditional variances.

Concavity of the value functions in the uniform model

We first break symmetry in the conventional way and take F to be the uniform distribution on $[0, 1]$. Specialization of the Bellman recursion (11) defines the sequence of value functions $\{v_k^u : 1 \le k < \infty\}$, and specialization of the characterization (12) defines the sequence of threshold functions $\{h_k^u : 1 \le k < \infty\}$. Here, we have by (12) that $h_k^u(s) \le 1$ for all $s \in [0, 1]$ and $1 \le k < \infty$.

Lemma 15 (*Concavity of the Uniform Value Functions*)**.** *For each* $1 \le k < \infty$ *the value function* $v_k^u : [0, 1] \to \mathbb{R}^+$ *is concave.*

Proof. Proposition 14 tells us v_k^u is continuously differentiable on $(0, 1)$, and we prove concavity by showing that $s \mapsto (v_k^u)'(s)$ non-increasing on $(0, 1)$. We let $\mathcal{H}_k$ be the assertion

$$(v_k^u)'(s + \epsilon) \le (v_k^u)'(s) \quad \text{for all } s \in (0, 1) \text{ and } 0 < \epsilon < 1 - s,$$

and we argue by induction. For $k = 1$ we have $v_1^u(s) = 1 - s$, so $(v_1^u)'(s) = -1$ and $\mathcal{H}_1$ holds trivially.

Now, if we specialize the derivative recursion (24) to the uniform model we have

$$(v_k^u)'(s) = -1 + \{1 - h_k^u(s) + s\}(v_{k-1}^u)'(s), \quad \text{for } s \in (0, 1),$$

so if we assume that $\mathcal{H}_{k-1}$ holds then we have

$$\begin{aligned}(v_k^u)'(s + \epsilon) - (v_k^u)'(s) &= \{1 - h_k^u(s) + s\}\{(v_{k-1}^u)'(s + \epsilon) - (v_{k-1}^u)'(s)\} \\ &\quad + \{h_k^u(s) - h_k^u(s + \epsilon) + \epsilon\}(v_{k-1}^u)'(s + \epsilon). \end{aligned} \tag{25}$$

Since $0 \le s \le h_k^u(s) \le 1$, we see from $\mathcal{H}_{k-1}$ that the first summand on the right-hand side of (25) is non-positive. Monotonicity of v_k^u also tells us $(v_{k-1}^u)'(s + \epsilon) \le 0$, so to complete the induction step we just need to check that

$$g(s, \epsilon) \stackrel{\text{def}}{=} h_k^u(s + \epsilon) - h_k^u(s) \le \epsilon. \tag{26}$$

From the definition of the critical value s_{k-1}^* we have

$$g(s, \epsilon) = \begin{cases} h_k^u(s + \epsilon) - h_k^u(s) & \text{if } s < s + \epsilon < s_{k-1}^* \\ 1 - h_k^u(s) & \text{if } s < s_{k-1}^* \le s + \epsilon \\ 0 & \text{if } s_{k-1}^* \le s < s + \epsilon, \end{cases}$$

so we only need to check (26) in the first two cases.

For $s < s+\epsilon < s^*_{k-1}$, we know by Lemma 13 that $h^u_k(s)$ is differentiable at s, so by the induction assumption $\mathcal{H}_{k-1}$ and the negativity of $(v^u_{k-1})'(s)$ we have

$$0 \le (h^u_k)'(s) \le 1 \quad \text{for all } s \in (0, s^*_{k-1}).$$

Thus, h^u_k is Lipschitz-1 continuous on $(0, s^*_{k-1})$, and we have

$$g(s,\epsilon) = h^u_k(s+\epsilon) - h^u_k(s) \le \epsilon \quad \text{for all } s < s+\epsilon < s^*_{k-1}.$$

For the second case where $s < s^*_{k-1} \le s+\epsilon$, we first note that $h^u_k(s^*_{k-1}) = 1$ and that h^u_k is continuous, so we have

$$g(s,\epsilon) = \lim_{u \nearrow s^*_{k-1}} \{h^u_k(u) - h^u_k(s)\} \le \lim_{u \nearrow s^*_{k-1}} \{u - s\} = s^*_{k-1} - s < \epsilon,$$

where the inequality follows from the Lipschitz-1 property of h^u_k. This completes the second check and the proof of the induction step. □

Convexity of the value functions in the exponential model

We now break the symmetry in a second way. We take $F(x) = 1 - e^{-x}$ for $x \ge 0$, and we let v^e_k and h^e_k denote the corresponding value and threshold functions. We will shortly find that v^e_k is convex on $[0,\infty)$ for all $k \ge 1$, but we need a preliminary lemma.

Lemma 16. *For $1 \le k < \infty$ and $s \in (0,\infty)$ one has*

$$-\{1 - e^{-h^e_{k+1}(s)+s}\}^{-1} \le (v^e_k)'(s). \tag{27}$$

Proof. By Proposition 14 we know that v^e_k is continuously differentiable and by (24) we have

$$(v^e_k)'(s) = (1 - e^{-s} + e^{-h^e_k(s)})(v^e_{k-1})'(s) - e^{-s}. \tag{28}$$

We now let $\mathcal{H}_k$ be the assertion that

$$-\{1 - e^{-h^e_{k+1}(s)+s}\}^{-1} \le (v^e_k)'(s), \quad \text{for all } s \in (0,\infty),$$

and we argue by induction. For $k = 1$ we have $v^e_1(s) = e^{-s} < 1$, so by (12) we have $h^e_2(s) = \infty$. In turn this gives us

$$-\{1 - e^{-h^e_2(s)+s}\}^{-1} = -1 \le -e^{-s} = (v^e_1)'(s),$$

which verifies $\mathcal{H}_1$.

Next, if we assume that $\mathcal{H}_{k-1}$ holds and we substitute the lower bound from $\mathcal{H}_{k-1}$ into (28), then rearrangement gives us

$$-\{1 - e^{-h^e_k(s)+s}\}^{-1}[1 - e^{-s}\{1 - e^{-h^e_k(s)+s}\}] - e^{-s} \le (v^e_k)'(s).$$

From (19) we have $h^e_{k+1}(s) \le h^e_k(s)$, so we now have

$$-\{1 - e^{-h^e_{k+1}(s)+s}\}^{-1} \le -\{1 - e^{-h^e_k(s)+s}\}^{-1} \le (v^e_k)'(s),$$

and this is just what one needs to complete the induction step. □

We now have the main result of this section.

Lemma 17 (*Convexity of the Exponential Value Functions*)**.** *For each* $1 \le k < \infty$*, the value function* $v_k^e : [0,\infty) \to \mathbb{R}^+$ *is convex on* $[0,\infty)$.

Proof. By Proposition 14 we know that v_k^e is continuously differentiable, and we again argue by induction. This time we take $\mathcal{H}_k$ to be the assertion

$$(v_k^e)'(s) \le (v_k^e)'(s+\epsilon) \quad \text{for all } s \in (0,\infty) \text{ and } \epsilon > 0.$$

For $k = 1$, we have $v_1^e(s) = e^{-s}$ and $(v_1^e)'(s) = -e^{-s}$ so the base case $\mathcal{H}_1$ of the induction is valid.

Now, by (28) applied twice we have

$$\begin{aligned}(v_k^e)'(s) - (v_k^e)'(s+\epsilon) = {} & [1 - e^{-s-\epsilon} + e^{-h_k^e(s+\epsilon)}]\{(v_{k-1}^e)'(s) - (v_{k-1}^e)'(s+\epsilon)\} \\ & + \{e^{-h_k^e(s)}[1 - e^{-h_k^e(s+\epsilon)+h_k^e(s)}] - e^{-s}[1 - e^{-\epsilon}]\}(v_{k-1}^e)'(s) \\ & - e^{-s}[1 - e^{-\epsilon}]. \end{aligned} \tag{29}$$

The induction hypothesis $\mathcal{H}_{k-1}$, tells us that $s \mapsto v_{k-1}^e(s)$ is convex, so by (23) we have $(h_k^e)'(s) \ge 1$ for $s \in (0,\infty)$, and this gives us the bound

$$-h_k^e(s+\epsilon) + h_k^e(s) \le -\epsilon.$$

We always have $s \le h_k^e(s)$ and $(v_{k-1}^e)'(s) \le 0$, so (29) implies the simpler bound

$$\begin{aligned}(v_k^e)'(s) - (v_k^e)'(s+\epsilon) \le {} & [1 - e^{-s-\epsilon} + e^{-h_k^e(s+\epsilon)}]\{(v_{k-1}^e)'(s) - (v_{k-1}^e)'(s+\epsilon)\} \\ & - e^{-s}[1 - e^{-\epsilon}][1 - e^{-h_k^e(s)+s}](v_{k-1}^e)'(s) - e^{-s}[1 - e^{-\epsilon}]. \end{aligned}$$

We only need to check that this bound is non-positive. By the induction hypothesis $\mathcal{H}_{k-1}$ and $s + \epsilon \le h_k^e(s+\epsilon)$, we see the first term is non-positive. The bound (27) tells us

$$-[1 - e^{-h_k^e(s)+s}](v_{k-1}^e)'(s) \le 1,$$

so, when we replace $-[1 - e^{-h_k^e(s)+s}](v_{k-1}^e)'(s)$ with its upper bound, we also see that the second and the third terms sum to zero. This completes the proof of the induction step and of the lemma. □

6. Martingale relations and $L_n(\pi_n^*)$

One can represent $L_n(\pi_n^*)$ as a sum of functionals of a time non-homogeneous Markov chain. To see how this goes, we first set $M_0 = 0$ and then we define M_i recursively by

$$M_i = \begin{cases} M_{i-1} & \text{if } X_i \notin [M_{i-1}, h_{n-i+1}(M_{i-1})] \\ X_i & \text{if } X_i \in [M_{i-1}, h_{n-i+1}(M_{i-1})], \end{cases} \tag{30}$$

so, less formally, M_i is the maximum value of the elements of the subsequence that have been selected up to and including time i.

Since we accept X_i if and only if $X_i \in [M_{i-1}, h_{n-i+1}(M_{i-1})]$ and since $L_n(\pi_n^*)$ counts the number of the observations $X_1, X_2, \ldots, X_n$ that we accept, we have

$$L_n(\pi_n^*) = \sum_{i=1}^{n} \mathbb{1}(X_i \in [M_{i-1}, h_{n-i+1}(M_{i-1})]). \tag{31}$$

It is also useful to set $L_0(\pi_n^*) = 0$ and to introduce the shorthand,

$$L_i(\pi_n^*) \stackrel{\text{def}}{=} \sum_{j=1}^{i} \mathbb{1}(X_j \in [M_{j-1}, h_{n-j+1}(M_{j-1})]), \quad \text{for } 1 \le i \le n.$$

We now come to a martingale that is central to the rest of our analysis.

Proposition 18 (*Optimality Martingale*)**.** *The process* $\{Y_i : i = 0, 1, \ldots, n\}$ *defined by setting*

$$Y_i = L_i(\pi_n^*) + v_{n-i}(M_i) \quad \textit{for } 0 \le i \le n, \tag{32}$$

is a martingale with respect to the filtration $\mathcal{F}_i = \sigma\{X_1, X_2, \ldots, X_i\}$, $1 \le i \le n$.

Proof. Obviously Y_i is $\mathcal{F}_i$-measurable and bounded. Moreover, by the definition of $v_{n-i}(s)$ we have $v_{n-i}(M_i) = \mathbb{E}[L_n(\pi_n^*) - L_i(\pi_n^*) \mid \mathcal{F}_i]$, so

$$Y_i = L_i(\pi_n^*) + \mathbb{E}[L_n(\pi_n^*) - L_i(\pi_n^*) \mid \mathcal{F}_i] = \mathbb{E}[L_n(\pi_n^*) \mid \mathcal{F}_i]. \quad \square$$

Since the martingale $\{Y_i : 1 \le i \le n\}$ is capped by $L_n(\pi_n^*)$, we also have the explicit identity

$$\mathbb{E}[L_n(\pi_n^*) \mid \mathcal{F}_i] = L_i(\pi_n^*) + v_{n-i}(M_i), \tag{33}$$

which is often useful.

Conditional variances

In (32), the term $v_{n-i}(M_i) = \mathbb{E}[L_n(\pi_n^*) - L_i(\pi_n^*) \mid \mathcal{F}_i]$ tells us the expected number of selections that the policy π_n^* will make from $\{X_{i+1}, X_{i+2}, \ldots, X_n\}$ given the current value M_i of the running maximum. There is a useful notion of *conditional variance* that is perfectly analogous. Specifically, we set

$$\begin{aligned} w_{n-i}(M_i) &\stackrel{\text{def}}{=} \operatorname{Var}[L_n(\pi_n^*) - L_i(\pi_n^*) \mid \mathcal{F}_i] \\ &= \mathbb{E}[\{L_n(\pi_n^*) - L_i(\pi_n^*) - v_{n-i}(M_i)\}^2 \mid \mathcal{F}_i]. \end{aligned} \tag{34}$$

Here, of course, if $i = 0$ we always have $M_0 = 0$ and

$$w_n(M_0) = \operatorname{Var}[L_n(\pi_n^*)].$$

The martingale $\{Y_i, \mathcal{F}_i\}_{i=0}^n$ defined by (32) leads in a natural way to an informative representation for the conditional variance (34), and one starts with the difference sequence

$$d_j = Y_j - Y_{j-1}, \quad \text{where } 1 \le j \le n. \tag{35}$$

By (32) and telescoping of the sum we have

$$\sum_{j=i+1}^{n} d_j = L_n(\pi_n^*) - L_i(\pi_n^*) - v_{n-i}(M_i), \quad \text{for } 0 \le i \le n, \tag{36}$$

so by orthogonality of the martingale differences we get

$$w_{n-i}(M_i) = \operatorname{Var}[L_n(\pi_n^*) - L_i(\pi_n^*) \mid \mathcal{F}_i] = \sum_{j=i+1}^{n} \mathbb{E}[d_j^2 \mid \mathcal{F}_i]. \tag{37}$$

This representation for the conditional variance w_{n-i} can be usefully reframed by taking a more structured view of the martingale differences (35). Specifically, we write

$$d_j = A_j + B_j, \tag{38}$$

where the variable

$$B_j \stackrel{\text{def}}{=} v_{n-j}(M_{j-1}) - v_{n-j+1}(M_{j-1}) \tag{39}$$

represents the change in the martingale Y_j when we do not select X_j, and where

$$A_j \stackrel{\text{def}}{=} (1 + v_{n-j}(X_j) - v_{n-j}(M_{j-1}))\mathbb{1}(X_j \in [M_{j-1}, h_{n-j+1}(M_{j-1})]) \tag{40}$$

is the *additional* contribution to the change in the martingale Y_j when we do select X_j. Since B_j is $\mathcal{F}_{j-1}$-measurable, we have

$$\mathbb{E}[d_j^2 \mid \mathcal{F}_{j-1}] = \mathbb{E}[A_j^2 \mid \mathcal{F}_{j-1}] + 2B_j\, \mathbb{E}[A_j \mid \mathcal{F}_{j-1}] + B_j^2,$$

and we also have $0 = \mathbb{E}[d_j \mid \mathcal{F}_{j-1}] = B_j + \mathbb{E}[A_j \mid \mathcal{F}_{j-1}]$, so

$$\mathbb{E}[d_j^2 \mid \mathcal{F}_{j-1}] = \mathbb{E}[A_j^2 \mid \mathcal{F}_{j-1}] - B_j^2. \tag{41}$$

Now, for $j = i+1$ to n, we take the conditional expectation in (41) with respect to $\mathcal{F}_i$. When we sum these terms and recall (37) we get our final representation for conditional variances

$$w_{n-i}(M_i) = \mathrm{Var}[L_n(\pi_n^*) - L_i(\pi_n^*) \mid \mathcal{F}_i] = \sum_{j=i+1}^{n} \{\mathbb{E}[A_j^2 \mid \mathcal{F}_i] - \mathbb{E}[B_j^2 \mid \mathcal{F}_i]\}. \tag{42}$$

The decomposition (42) was our main goal here, but before concluding the section we should make one further inference from (38). By the representation (12) for h_k we have $0 \le A_j \le 1$, and by our bound (15) on the value function differences we have $-1 \le B_j \le 0$. Hence one has a uniform bound on the martingale differences

$$|d_j| = |A_j + B_j| \le 1 \quad \text{for } 1 \le j \le n. \tag{43}$$

7. Inferences from the uniform model

We now consider the decompositions of Section 6 when F is the uniform distribution on $[0, 1]$, and we use superscripts to make this specialization explicit. In particular, we let $X_1^u, X_2^u, \ldots, X_n^u$ be the underlying sequence of n independent uniformly distributed random variables, and we write M_i^u for the value of the last observation selected up to and including time $i \ge 1$ (and, as usual, we set $M_0^u = 0$). Lemma 15 tells us that the value function v_k^u is concave, and this is crucial to the proof of the lower bound for the conditional variance of $L_n^u(\pi_n^*)$.

Proposition 19 (*Conditional Variance Lower Bound*)**.** *For $0 \le i \le n$ one has*

$$\frac{1}{3}\, v_{n-i}^u(M_i^u) - 2 \le w_{n-i}^u(M_i^u).$$

Proof. Specialization of the representation (42) gives us

$$w^u_{n-i}(M^u_i) = \sum_{j=i+1}^{n} \mathbb{E}[(A^u_j)^2 \mid \mathcal{F}_i] - \sum_{j=i+1}^{n} \mathbb{E}[(B^u_j)^2 \mid \mathcal{F}_i], \tag{44}$$

where the definitions (39) and (40) now become

$$B^u_j = v^u_{n-j}(M^u_{j-1}) - v^u_{n-j+1}(M^u_{j-1})$$

and

$$A^u_j = (1 + v^u_{n-j}(X_j) - v^u_{n-j}(M^u_{j-1}))\mathbb{1}(X^u_j \in [M^u_{j-1}, h^u_{n-j+1}(M^u_{j-1})]). \tag{45}$$

First, we work toward a lower bound for the leading sum in (44). If we square both sides of (45) and take conditional expectations, then we have

$$\mathbb{E}[(A^u_j)^2 \mid \mathcal{F}_{j-1}] = \int_{M^u_{j-1}}^{h^u_{n-j+1}(M^u_{j-1})} \{1 + v^u_{n-j}(x) - v^u_{n-j}(M^u_{j-1})\}^2 \, dx. \tag{46}$$

By Lemma 15 the map $x \mapsto v^u_{n-j}(x)$ is concave in x, so the line through the points $(M^u_{j-1}, 1)$ and $(h^u_{n-j+1}(M^u_{j-1}), 0)$ provides a lower bound on the integrand in (46). Integration of this linear lower bound then gives

$$\frac{1}{3}\left(h^u_{n-j+1}(M^u_{j-1}) - M^u_{j-1}\right) \le \int_{M^u_{j-1}}^{h^u_{n-j+1}(M^u_{j-1})} \{1 + v^u_{n-j}(x) - v^u_{n-j}(M^u_{j-1})\}^2 \, dx. \tag{47}$$

From the definition of $L^u_i(\pi^*_n)$ we have the identity

$$\sum_{j=i+1}^{n} \mathbb{E}[h^u_{n-j+1}(M^u_{j-1}) - M^u_{j-1} \mid \mathcal{F}_i] = \mathbb{E}[L^u_n(\pi^*_n) - L^u_i(\pi^*_n) \mid \mathcal{F}_i] = v^u_{n-i}(M^u_i),$$

so (46) and (47) give us

$$\frac{1}{3} v^u_{n-i}(M^u_i) \le \sum_{j=i+1}^{n} \mathbb{E}[(A^u_j)^2 \mid \mathcal{F}_i]. \tag{48}$$

Now, to work toward an upper bound on $\mathbb{E}[(B^u_j)^2 \mid \mathcal{F}_i]$, we first note by the crude Lemma 5 that

$$(B^u_j)^2 = \left(v^u_{n-j}(M^u_{j-1}) - v^u_{n-j+1}(M^u_{j-1})\right)^2 \le \left(h^u_{n-j+1}(M^u_{j-1}) - M^u_{j-1}\right)^2. \tag{49}$$

The definition (30) of the running maximum M^u_j, the uniform distribution of X^u_j, and calculus give us the identity

$$\begin{aligned} \mathbb{E}[M^u_j - M^u_{j-1} \mid \mathcal{F}_{j-1}] &= \int_{M^u_{j-1}}^{h^u_{n-j+1}(M^u_{j-1})} (x - M^u_{j-1}) \, dx \\ &= \frac{1}{2}\left(h^u_{n-j+1}(M^u_{j-1}) - M^u_{j-1}\right)^2, \end{aligned} \tag{50}$$

so (49) gives us the succinct bound

$$(B^u_j)^2 \le 2\,\mathbb{E}[M^u_j - M^u_{j-1} \mid \mathcal{F}_{j-1}].$$

Now we take the conditional expectation with respect to $\mathcal{F}_i$ and sum over $i < j \le n$. Telescoping then gives us

$$\sum_{j=i+1}^{n} \mathbb{E}[(B_j^u)^2 \mid \mathcal{F}_i] \le 2\,\mathbb{E}[M_n^u - M_i^u \mid \mathcal{F}_i] \le 2, \tag{51}$$

where, in the last step, we used $0 \le M_i^u \le M_n^u \le 1$. The representation (44) and the bounds (48) and (51) complete the proof of the lemma. □

A Cauchy–Schwarz argument

If we take the total expectation in (50), then we have

$$\mathbb{E}[\left(h_{n-j+1}^u(M_{j-1}^u) - M_{j-1}^u\right)^2] = 2\{\mathbb{E}[M_j^u] - \mathbb{E}[M_{j-1}^u]\}, \tag{52}$$

and, since $\mathbb{E}[(h_{n-j+1}^u(M_{j-1}^u) - M_{j-1}^u)]$ is the unconditional probability that we accept the j'th element of the sequence, one might hope to estimate $\mathbb{E}[L_n(\pi_n^*)]$ with help from (52) and a Cauchy–Schwarz argument.

In fact, by two applications of the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
\mathbb{E}[L_n^u(\pi_n^*)] &= \sum_{j=1}^{n} \mathbb{E}[h_{n-j+1}^u(M_{j-1}^u) - M_{j-1}^u] \\
&\le n^{1/2} \left\{ \sum_{j=1}^{n} (\mathbb{E}[h_{n-j+1}^u(M_{j-1}^u) - M_{j-1}^u])^2 \right\}^{1/2} \\
&\le n^{1/2} \left\{ \sum_{j=1}^{n} \mathbb{E}\left[\left(h_{n-j+1}^u(M_{j-1}^u) - M_{j-1}^u\right)^2 \right] \right\}^{1/2},
\end{aligned}$$

and, when we replace all of the summands using (52), we get a telescoping sum

$$\mathbb{E}[L_n^u(\pi_n^*)] \le n^{1/2} \left\{ 2 \sum_{j=1}^{n} \{\mathbb{E}[M_j^u] - \mathbb{E}[M_{j-1}^u]\} \right\}^{1/2} = (2n)^{1/2} \{\mathbb{E}[M_n^u]\}^{1/2}.$$

We have $\mathbb{E}[M_n^u] < 1$ since the support of M_n^u equals $[0, 1]$, and, since the distribution of $L_n^u(\pi_n^*)$ does not depend on F, we find for all continuous F that

$$\mathbb{E}[L_n(\pi_n^*)] < (2n)^{1/2}. \tag{53}$$

This recaptures the mean upper bound (3) of Bruss and Robertson [10] and Gnedin [12] which was discussed in the introduction.

Here we should note that Bruss and Delbaen [8] also used a Cauchy–Schwarz argument to show that for Poisson N_ν with mean ν, one has the analogous inequality

$$\mathbb{E}[L_{N_\nu}(\pi_{N_\nu}^*)] \le (2\nu)^{1/2}. \tag{54}$$

We know by Proposition 11 that for $\nu = n$ we have $\mathbb{E}[L_{N_n}(\pi_{N_n}^*)] \le \mathbb{E}[L_n(\pi_n^*)]$ but, even so, the bound (54) does not help directly with (53)—or vice versa. In addition to the usual issue that "policies do not de-Poissonize", there is the real *a priori* possibility that $\mathbb{E}[L_{N_n}(\pi_{N_n}^*)]$ might be much smaller than $\mathbb{E}[L_n(\pi_n^*)]$.

8. Inferences from the exponential model

Now we consider the exponential distribution $F(x) = 1 - e^{-x}$, for $x \geq 0$, and, as before, we use superscripts to make this specialization explicit. Thus $X_1^e, X_2^e, \ldots, X_n^e$ denotes a sequence of n independent, mean one, exponential random variables, and M_i^e denotes the value of the last observation selected up to and including time $i \geq 1$ (and, again, we set $M_0^e = 0$). This time Lemma 17 provides the critical fact; it tells us that the value function v_k^e is convex, and this is at the heart of the argument.

Proposition 20 (*Conditional Variance Upper Bound*)**.** *For each* $0 \leq i \leq n$ *one has*

$$w_{n-i}^e(M_i^e) \leq \frac{1}{3}\, v_{n-i}^e(M_i^e) + \frac{2}{3}\{1 + \log(n-i)\}. \tag{55}$$

The proof roughly parallels that of Proposition 19, but in this case some integrals are more troublesome to estimate. To keep the argument direct, we extract one calculation as a lemma.

Lemma 21. *For* $0 \leq s < t < \infty$ *one has*

$$\int_s^t \left(\frac{t-x}{t-s}\right)^2 e^{-x}\, dx \leq \frac{1}{3}(e^{-s} - e^{-t}) + \frac{2}{3}\{e^{-s} - e^{-t}(t-s+1)\}.$$

Proof. If we set

$$g(y) \stackrel{\text{def}}{=} y^{-2}\{-6y + e^{-y}(2y^3 + 3y^2 - 6 + 6e^y)\} \quad \text{for } y \geq 0,$$

then by integration and simplification one has for $0 \leq s < t < \infty$ that

$$\frac{e^{-s}}{3} g(t-s) = \int_s^t \left(\frac{t-x}{t-s}\right)^2 e^{-x}\, dx - \frac{1}{3}(e^{-s} - e^{-t}) - \frac{2}{3}\{e^{-s} - e^{-t}(t-s+1)\},$$

and the lemma follows if we verify that $g(y) \leq 0$ for all $y \geq 0$. By the integral representation

$$g(y) = y^{-2}\left(\int_0^y (-6)\, dx + \int_0^y e^{-x}(6 + 6x + 3x^2 - 2x^3)\, dx\right),$$

we see that it suffices to show that

$$6 + 6x + 3x^2 - 2x^3 \leq 6e^x \quad \text{for all } x \in [0, \infty),$$

and the last inequality is obvious from the power series of e^x. □

Proof of Proposition 20. Specialization of (42) to the exponential model and simplification give us

$$w_{n-i}^e(M_i^e) = \mathrm{Var}[L_n^e(\pi_n^*) - L_i^e(\pi_n^*) \mid \mathcal{F}_i] \leq \sum_{j=i+1}^{n} \mathbb{E}[(A_j^e)^2 \mid \mathcal{F}_i], \tag{56}$$

where A_j^e is given by

$$A_j^e = (1 + v_{n-j}^e(X_j^e) - v_{n-j}^e(M_{j-1}^e))\mathbb{1}(X_j^e \in [M_{j-1}^e, h_{n-j+1}^e(M_{j-1}^e)]).$$

Since M^e_{j-1} is $\mathcal{F}_{j-1}$-measurable, we have

$$\mathbb{E}[(A^e_j)^2 \mid \mathcal{F}_{j-1}] = \int_{M^e_{j-1}}^{h^e_{n-j+1}(M^e_{j-1})} \{1 + v^e_{n-j}(x) - v^e_{n-j}(M^e_{j-1})\}^2 e^{-x}\, dx, \tag{57}$$

and by Lemma 17 the map $x \mapsto 1 + v^e_{n-j}(x) - v^e_{n-j}(M^e_{j-1})$ is convex in x and non-negative for all $x \in [M^e_{j-1}, h^e_{n-j+1}(M^e_{j-1})]$.

If $h^e_{n-j+1}(M^e_{j-1}) < \infty$, the line through the left-end point $(M^e_{j-1}, 1)$ and the right-end point $(h^e_{n-j+1}(M^e_{j-1}), 0)$ provides us with an easy upper bound for the integrand (57). Specifically, for $x \in [M^e_{j-1}, h^e_{n-j+1}(M^e_{j-1})]$, we have that

$$\{1 + v^e_{n-j}(x) - v^e_{n-j}(M^e_{j-1})\}^2 e^{-x} \le \left(\frac{h^e_{n-j+1}(M^e_{j-1}) - x}{h^e_{n-j+1}(M^e_{j-1}) - M^e_{j-1}} \right)^2 e^{-x}. \tag{58}$$

On the other hand, if $h^e_{n-j+1}(M^e_{j-1}) = \infty$, the right-side of (58) is replaced by e^{-x}, and (58) again holds since $0 \le \{1 + v^e_{n-j}(x) - v^e_{n-j}(M^e_{j-1})\} \le 1$.

We now integrate (58) and use the bound of Lemma 21; the representation (57) then gives us

$$\begin{aligned} \mathbb{E}[(A^e_j)^2 \mid \mathcal{F}_{j-1}] \le{}& \frac{1}{3}(e^{-M^e_{j-1}} - e^{-h^e_{n-j+1}(M^e_{j-1})}) \\ &+ \frac{2}{3}\{e^{-M^e_{j-1}} - e^{-h^e_{n-j+1}(M^e_{j-1})}(1 + h^e_{n-j+1}(M^e_{j-1}) - M^e_{j-1})\}. \end{aligned} \tag{59}$$

Now we just need to interpret the two addends on the right-hand side of (59). The first addend is just the probability that observation X^e_j is selected when the value of the running maximum is M^e_{j-1}, that is,

$$\mathbb{E}[\mathbb{1}(X^e_j \in [M^e_{j-1}, h^e_{n-j+1}(M^e_{j-1})]) \mid \mathcal{F}_{j-1}] = e^{-M^e_{j-1}} - e^{-h^e_{n-j+1}(M^e_{j-1})}. \tag{60}$$

Similarly, the second addend of (59) is the one-period expected increment of the current running maximum M^e_{j-1}, or, to be explicit,

$$\begin{aligned} \mathbb{E}[M^e_j - M^e_{j-1} \mid \mathcal{F}_{j-1}] &= \int_{M^e_{j-1}}^{h^e_{n-j+1}(M^e_{j-1})} (x - M^e_{j-1}) e^{-x}\, dx \\ &= e^{-M^e_{j-1}} - e^{-h^e_{n-j+1}(M^e_{j-1})}(1 + h^e_{n-j+1}(M^e_{j-1}) - M^e_{j-1}). \end{aligned} \tag{61}$$

Given the two interpretations (60) and (61), our bound (59) now becomes

$$\begin{aligned} \mathbb{E}[(A^e_j)^2 \mid \mathcal{F}_{j-1}] \le{}& \frac{1}{3}\, \mathbb{E}[\mathbb{1}(X^e_j \in [M^e_{j-1}, h^e_{n-j+1}(M^e_{j-1})]) \mid \mathcal{F}_{j-1}] \\ &+ \frac{2}{3}\, \mathbb{E}[M^e_j - M^e_{j-1} \mid \mathcal{F}_{j-1}]. \end{aligned}$$

Next, we recall the variance upper bound (56), take conditional expectations with respect to $\mathcal{F}_i$, and sum over $i + 1 \le j \le n$, to obtain

$$w^e_{n-i}(M^e_i) \le \frac{1}{3}\,\mathbb{E}[L^e_n(\pi^*_n) - L^e_i(\pi^*_n) \mid \mathcal{F}_i] + \frac{2}{3}\,\mathbb{E}[M^e_n - M^e_i \mid \mathcal{F}_i]$$
$$= \frac{1}{3}\,v^e_{n-i}(M^e_i) + \frac{2}{3}\,\mathbb{E}[M^e_n - M^e_i \mid \mathcal{F}_i], \tag{62}$$

where in the last step we used the martingale identity (33).

To complete the proof, we only need to estimate the conditional expectation $\mathbb{E}[M^e_n - M^e_i \mid \mathcal{F}_i]$. We first set $\mathcal{M}^*_{[i+1,n]} = \max\{X^e_{i+1}, X^e_{i+2}, \dots, X^e_n\}$, and then we note that

$$M^e_n - M^e_i \le \max\{\mathcal{M}^*_{[i+1,n]}, M^e_i\} - M^e_i \le \mathcal{M}^*_{[i+1,n]}.$$

When we take the conditional expectations and use the independence of $\mathcal{F}_i$ and $\{X^e_{i+1}, X^e_{i+2}, \dots, X^e_n\}$, we get

$$\mathbb{E}[M^e_n - M^e_i \mid \mathcal{F}_i] \le \mathbb{E}[\mathcal{M}^*_{[i+1,n]} \mid \mathcal{F}_i] = \mathbb{E}[\mathcal{M}^*_{[1,n-i]}]. \tag{63}$$

The logarithmic bound for the last term is well-known, but, for completeness, we just note $\mathbb{P}(\mathcal{M}^*_{[1,n-i]} \le t) = (1 - e^{-t})^{n-i}$, so

$$\mathbb{E}[\mathcal{M}^*_{[1,n-i]}] = \int_0^\infty 1 - (1 - e^{-t})^{n-i}\,dt = \sum_{j=1}^{n-i} j^{-1} \le 1 + \log(n-i).$$

This last estimate then combines with the upper bounds (62) and (63) to complete the proof of (55). □

9. Combined inferences: variance bounds in general

The variance bounds obtained under the uniform and exponential models are almost immediately applicable to general continuous F. One only needs to make an appropriate translation.

Proposition 22. *For any continuous F and for all $0 \le i \le n$ one has the conditional variance bounds*

$$\frac{1}{3}\,v_{n-i}(M_i) - 2 \le w_{n-i}(M_i) \le \frac{1}{3}\,v_{n-i}(M_i) + \frac{2}{3}\{1 + \log(n-i)\}. \tag{64}$$

In particular, for $i = 0$ one has $M_0 = 0$ and

$$\frac{1}{3}\,\mathbb{E}[L_n(\pi^*_n)] - 2 \le \mathrm{Var}[L_n(\pi^*_n)] \le \frac{1}{3}\,\mathbb{E}[L_n(\pi^*_n)] + \frac{2}{3}\{1 + \log n\}. \tag{65}$$

Proof. If $X_1, X_2, \dots, X_n$ is a sequence of independent random variables with the continuous distribution F, then the familiar transformations

$$X^u_i \stackrel{\text{def}}{=} F(X_i) \quad \text{and} \quad X^e_i \stackrel{\text{def}}{=} -\log\{1 - F(X_i)\}$$

define sequences that have the uniform and exponential distribution, respectively. These transformations give us a dictionary that we can use to translate results between our models; specifically we have:

$$v_k(s) = v^u_k(F(s)) \quad \text{and} \quad v_k(s) = v^e_k(-\log\{1 - F(s)\}),$$
$$M^u_i = F(M_i) \quad \text{and} \quad M^e_i = -\log\{1 - F(M_i)\},$$
$$w_{n-i}(M_i) = w^u_{n-i}(M^u_i) \quad \text{and} \quad w_{n-i}(M_i) = w^e_{n-i}(M^e_i).$$

Proposition 19 tells us that

$$\frac{1}{3}\, v^u_{n-i}(M^u_i) - 2 \le w^u_{n-i}(M^u_i),$$

so the first column of the dictionary gives us the first inequality of (64). Similarly, Proposition 20 tells us

$$w^e_{n-i}(M^e_i) \le \frac{1}{3}\, v^e_{n-i}(M^e_i) + \frac{2}{3}\{1 + \log(n-i)\},$$

and the second column of the dictionary gives us the second inequality of (64). □

10. The central limit theorem

Our proof of the central limit theorem for $L_n(\pi^*_n)$ depends on the most basic version of the martingale central limit theorem. Brown [7], McLeish [20], and Hall and Heyde [13] all give variations containing this one.

Proposition 23 (*Martingale Central Limit Theorem*)**.** *For each $n \ge 1$, we consider a martingale difference sequence $\{Z_{n,j} : 1 \le j \le n\}$ with respect to the sequence of increasing σ-fields $\{\mathcal{F}_{n,j} : 0 \le j \le n\}$. If*

$$\max_{1 \le j \le n} \|Z_{n,j}\|_\infty \to 0 \quad \text{as } n \to \infty \tag{66}$$

and

$$\sum_{j=1}^{n} \mathbb{E}[Z^2_{n,j} | \mathcal{F}_{j-1}] \xrightarrow{\text{p}} 1 \quad \text{as } n \to \infty, \tag{67}$$

then we have the convergence in distribution

$$\sum_{j=1}^{n} Z_{n,j} \Longrightarrow N(0,1) \quad \text{as } n \to \infty.$$

For each $n \ge 1$, we consider a driving sequence $X_{n,1}, X_{n,2}, \ldots, X_{n,n}$ of independent random variables with the continuous distribution F. We then set

$$Z_{n,j} \stackrel{\text{def}}{=} \frac{3^{1/2} d_{n,j}}{(2n)^{1/4}}, \quad \text{for } 1 \le j \le n,$$

where the $d_{n,j}$'s are the differences defined by (35), although we now make explicit the dependence of the differences on n. This is a martingale difference sequence with respect to the increasing sequence of σ-fields $\mathcal{F}_{n,j} = \sigma\{X_{n,1}, X_{n,2}, \ldots, X_{n,j}\}$, and when we take $i = 0$ in (36) we get the basic representation

$$\sum_{j=1}^{n} Z_{n,j} = \frac{3^{1/2}\{L_n(\pi^*_n) - \mathbb{E}[L_n(\pi^*_n)]\}}{(2n)^{1/4}}.$$

We know from (43) that we always have $|d_{n,j}| \le 1$ so, by our normalization, the negligibility condition (66) is trivially valid. The heart of the matter is the proof of the weak law (67); more

explicitly, we need to show that

$$\sum_{j=1}^{n} \frac{3\,\mathbb{E}[d_{n,j}^2 \mid \mathcal{F}_{n,j-1}]}{(2n)^{1/2}} \xrightarrow{\text{p}} 1 \quad \text{as } n \to \infty. \tag{68}$$

The variance bounds (65) and the asymptotic relation (2) for the mean imply

$$\text{Var}[L_n(\pi_n^*)] \sim \frac{1}{3}\mathbb{E}[L_n(\pi_n^*)] \sim \frac{(2n)^{1/2}}{3} \quad \text{as } n \to \infty,$$

and telescoping and orthogonality of the differences $\{d_{n,j} : 1 \le j \le n\}$ give us

$$\text{Var}[L_n(\pi_n^*)] = \mathbb{E}\left[\sum_{j=1}^{n} \mathbb{E}[d_{n,j}^2]\right] = \mathbb{E}\left[\sum_{j=1}^{n} \mathbb{E}[d_{n,j}^2 \mid \mathcal{F}_{n,j-1}]\right],$$

so the weak law (68) will follow from Chebyshev's inequality if one proves that

$$\text{Var}\left[\sum_{j=1}^{n} \mathbb{E}[d_{n,j}^2 \mid \mathcal{F}_{n,j-1}]\right] = o(n) \quad \text{as } n \to \infty. \tag{69}$$

The proof of Theorem 1 is completed once one confirms the relation (69), and the next lemma gives us more than we need.

Lemma 24 (*Conditional Variance Bound*)**.** *If F is continuous, then for $n \ge 1$, one has*

$$\text{Var}\left[\sum_{j=1}^{n} \mathbb{E}[d_{n,j}^2 \mid \mathcal{F}_{n,j-1}]\right] \le \{18 + (\log n)^2\}(2n)^{1/2}.$$

Proof. We fix $n \ge 1$ and simplify the notation by dropping the subscript n on the martingale difference sequence and the filtration. We then let

$$V \stackrel{\text{def}}{=} \sum_{j=1}^{n} \mathbb{E}[d_j^2 \mid \mathcal{F}_{j-1}]$$

and consider the martingale $\{V_i : 0 \le i \le n\}$ defined by setting

$$V_i \stackrel{\text{def}}{=} \mathbb{E}[V \mid \mathcal{F}_i\,] \quad \text{for } 0 \le i \le n.$$

One has the initial and terminal values

$$V_0 = \sum_{j=1}^{n} \mathbb{E}[d_j^2] = \text{Var}[L_n(\pi_n^*)] \quad \text{and} \quad V_n = V,$$

and if we introduce the new martingale differences $\Delta_i = V_i - V_{i-1}$, $1 \le i \le n$, then telescoping and orthogonality give us

$$V_n - V_0 = \sum_{i=1}^{n} \Delta_i \quad \text{and} \quad \text{Var}[V_n] = \text{Var}\left[\sum_{j=1}^{n} \mathbb{E}[d_j^2 \mid \mathcal{F}_{j-1}]\right] = \sum_{i=1}^{n} \mathbb{E}[\Delta_i^2]. \tag{70}$$

For $1 \le j \le i+1$ all of the summands $\mathbb{E}[d_j^2 \mid \mathcal{F}_{j-1}]$ are $\mathcal{F}_i$-measurable, so we have

$$\begin{aligned}\Delta_i &= \sum_{j=1}^{i} \mathbb{E}[d_j^2 \mid \mathcal{F}_{j-1}] + \mathbb{E}\left[\sum_{j=i+1}^{n} \mathbb{E}[d_j^2 \mid \mathcal{F}_{j-1}] \mid \mathcal{F}_i\right] \\ &\quad - \sum_{j=1}^{i} \mathbb{E}[d_j^2 \mid \mathcal{F}_{j-1}] - \mathbb{E}\left[\sum_{j=i+1}^{n} \mathbb{E}[d_j^2 \mid \mathcal{F}_{j-1}] \mid \mathcal{F}_{i-1}\right].\end{aligned}$$

The first and the third sum cancel, and we obtain

$$\begin{aligned}\Delta_i &= \sum_{j=i+1}^{n} \mathbb{E}[d_j^2 \mid \mathcal{F}_i] - \mathbb{E}\left[\sum_{j=i+1}^{n} \mathbb{E}[d_j^2 \mid \mathcal{F}_i] \mid \mathcal{F}_{i-1}\right] \\ &= w_{n-i}(M_i) - \mathbb{E}[w_{n-i}(M_i) \mid \mathcal{F}_{i-1}], \end{aligned} \tag{71}$$

where in the last line we twice used the formula (37) for the conditional variance.

Next, we set

$$G_i \stackrel{\text{def}}{=} \{\omega : X_i(\omega) \in [M_{i-1}(\omega), h_{n-i+1}(M_{i-1}(\omega))]\},$$

so, in words, G_i is the set of all ω for which the observation $X_i(\omega)$ is selected at time i under the optimal policy π_n^*. By the recursive definition (30) of the running maximum M_i, we then have the decomposition

$$w_{n-i}(M_i) = w_{n-i}(M_{i-1}) + \{w_{n-i}(X_i) - w_{n-i}(M_{i-1})\}\mathbb{1}(G_i). \tag{72}$$

In fact, one can replace w_{n-i} with any function here, and it will be useful to also note that

$$v_{n-i}(M_i) = v_{n-i}(M_{i-1}) + \{v_{n-i}(X_i) - v_{n-i}(M_{i-1})\}\mathbb{1}(G_i). \tag{73}$$

The first summand on the right-hand side of (72) is $\mathcal{F}_{i-1}$-measurable, so, if we rewrite (71) using (72) we obtain

$$\Delta_i = \{w_{n-i}(X_i) - w_{n-i}(M_{i-1})\}\mathbb{1}(G_i) - \mathbb{E}[\{w_{n-i}(X_i) - w_{n-i}(M_{i-1})\}\mathbb{1}(G_i) \mid \mathcal{F}_{i-1}].$$

When we square this identity and take the conditional expectation we find

$$\mathbb{E}[\Delta_i^2 \mid \mathcal{F}_{i-1}] \le \mathbb{E}[\{w_{n-i}(X_i) - w_{n-i}(M_{i-1})\}^2\mathbb{1}(G_i) \mid \mathcal{F}_{i-1}], \tag{74}$$

and all that remains is to estimate the difference $\{w_{n-i}(X_i) - w_{n-i}(M_{i-1})\}\mathbb{1}(G_i)$.

Now consider the *upper bound* in (64) and replace $w_{n-i}(M_i)$ and $v_{n-i}(M_i)$ with their decompositions (72) and (73). When we move the term $w_{n-i}(M_{i-1})$ to the right side, we have

$$\begin{aligned}\{w_{n-i}(X_i) - w_{n-i}(M_{i-1})\}\mathbb{1}(G_i) &\le \frac{1}{3}\{v_{n-i}(X_i) - v_{n-i}(M_{i-1})\}\mathbb{1}(G_i) \\ &\quad + \frac{1}{3}v_{n-i}(M_{i-1}) - w_{n-i}(M_{i-1}) + \frac{2}{3}(1+\log n). \end{aligned} \tag{75}$$

By the *lower bound* in (64) the second summand is bounded by two, and, to estimate the first summand, we note that the characterization (12) for the optimal threshold function and the monotonicity of the value function give us

$$\frac{1}{3}|\, v_{n-i}(X_i) - v_{n-i}(M_{i-1})\,|\mathbb{1}(G_i) \le \frac{1}{3}\mathbb{1}(G_i). \tag{76}$$

The left-hand side of (75) is zero off of the set G_i, so using (76) we see that (75) gives us

$$\{w_{n-i}(X_i) - w_{n-i}(M_{i-1})\}\mathbb{1}(G_i) \le \left\{3 + \frac{2}{3}\log n\right\}\mathbb{1}(G_i). \tag{77}$$

A parallel argument gives us the complementary inequality,

$$-\left\{3 + \frac{2}{3}\log n\right\}\mathbb{1}(G_i) \le \{w_{n-i}(X_i) - w_{n-i}(M_{i-1})\}\mathbb{1}(G_i). \tag{78}$$

Specifically, one now begins with the *lower bound* in (64) and replaces $w_{n-i}(M_i)$ and $v_{n-i}(M_i)$ with their decompositions.

Taken together (77) and (78) imply

$$|w_{n-i}(X_i) - w_{n-i}(M_{i-1})|\mathbb{1}(G_i) \le \left\{3 + \frac{2}{3}\log n\right\}\mathbb{1}(G_i),$$

so we can square both sides and take conditional expectations with respect to $\mathcal{F}_{i-1}$. By (74) and the definition of G_i, we then have

$$\mathbb{E}[\Delta_i^2 \mid \mathcal{F}_{i-1}] \le \left\{18 + \frac{8}{9}(\log n)^2\right\}\mathbb{E}[\mathbb{1}(X_i \in [M_{i-1}, h_{n-i+1}(M_{i-1})]) \mid \mathcal{F}_{i-1}],$$

so if we drop the factor $8/9$, take total expectations, and sum we get

$$\sum_{i=1}^{n}\mathbb{E}[\Delta_i^2] \le \{18 + (\log n)^2\}\sum_{i=1}^{n}\mathbb{E}[\mathbb{1}(X_i \in [M_{i-1}, h_{n-i+1}(M_{i-1})])]. \tag{79}$$

By (70) the sum on the left is the variance of $V_n = \sum_{j=1}^{n}\mathbb{E}[d_j^2 \mid \mathcal{F}_{j-1}]$, and by (31) the sum of the expected values on the right is equal to $\mathbb{E}[L_n(\pi_n^*)]$. Finally, we know that $\mathbb{E}[L_n(\pi_n^*)] < (2n)^{1/2}$ from (3) and the argument of Section 7, so (79) completes the proof of the lemma. □

11. Concluding observations

The idea of "spending symmetry" that was mentioned in Section 5 originates with an instructive essay of Tao [26, Section 1.4]. This notion can be cast in stunning generality, but here it turns out to be resolutely concrete and very useful.

The variance lower bound of (65) had been known to us for some years, but dogged analysis of the uniform model left us without an upper bound of comparable quality. A general Markov decision problem (MDP) bound in Arlotto, Gans and Steele [2] would give $\mathrm{Var}[L_n(\pi_n^*)] \le \mathbb{E}[L_n(\pi_n^*)]$, but here the MDP bound is too weak by a factor of three. It cannot serve even as good motivation for a central limit theorem.

With such a long tradition of immediate reduction to the uniform model, it was surprising to see how fruitful it could be to simultaneously use the exponential model—even though the distribution of $L_n(\pi_n^*)$ is the *same* under either model. Still, with different value functions come different qualitative features, and the convexity of the value functions under the exponential model leads in a natural way to the needed upper bound of the variance. This opened up the way to the rest of the analysis.

We mentioned one open problem earlier (see Remark 10), and there is a related problem that deserves some thought. In the offline selection problem, the distribution of the length of the longest increasing subsequence of a sequence of n independent uniformly distributed random

variables is the same as the distribution of the length of the longest increasing subsequence of a random permutation of the integers $\{1, 2, \ldots, n\}$. This equivalence is lost in the online selection problem, and it is unclear how much of Theorem 1 can be recaptured.

For example, if we write $L_n^{\text{perm}}(\pi_n^*)$ for the analog of $L_n(\pi_n^*)$ where now one chooses a random permutation of $\{1, 2, \ldots, n\}$, then, by an argument of Burgess Davis given in Samuels and Steele [25], one does have $\mathbb{E}[L_n^{\text{perm}}(\pi_n^*)] \sim (2n)^{1/2}$ as $n \to \infty$. Unfortunately, mean bounds like those of Theorem 1 cannot be achieved in this way, and variance bounds that would be good enough to support a central limit theorem are even more remote. Nevertheless, some analog of Theorem 1 is quite likely to be true.

References

[1] D. Aldous, P. Diaconis, Longest increasing subsequences: from patience sorting to the Baik–Deift–Johansson theorem, Bull. Amer. Math. Soc. (NS) 36 (4) (1999) 413–432.
[2] A. Arlotto, N. Gans, J.M. Steele, Markov decision problems where means bound variances, Oper. Res. 62 (4) (2014) 864–875.
[3] A. Arlotto, J.M. Steele, Optimal sequential selection of a unimodal subsequence of a random sequence, Combin. Probab. Comput. 20 (6) (2011) 799–814.
[4] J. Baik, P. Deift, K. Johansson, On the distribution of the length of the longest increasing subsequence of random permutations, J. Amer. Math. Soc. 12 (4) (1999) 1119–1178.
[5] B. Bollobás, G. Brightwell, The height of a random partial order: concentration of measure, Ann. Appl. Probab. 2 (4) (1992) 1009–1018.
[6] B. Bollobás, S. Janson, On the length of the longest increasing subsequence in a random permutation, in: Combinatorics, Geometry and Probability (Cambridge, 1993), Cambridge Univ. Press, Cambridge, 1997, pp. 121–128.
[7] B.M. Brown, Martingale central limit theorems, Ann. Math. Statist. 42 (1971) 59–66.
[8] F.T. Bruss, F. Delbaen, Optimal rules for the sequential selection of monotone subsequences of maximum expected length, Stochastic Process. Appl. 96 (2) (2001) 313–342.
[9] F.T. Bruss, F. Delbaen, A central limit theorem for the optimal selection process for monotone subsequences of maximum expected length, Stochastic Process. Appl. 114 (2) (2004) 287–311.
[10] F.T. Bruss, J.B. Robertson, "Wald's lemma" for sums of order statistics of i.i.d. random variables, Adv. Appl. Probab. 23 (3) (1991) 612–623.
[11] P. Flajolet, R. Sedgewick, Analytic Combinatorics, Cambridge University Press, Cambridge, 2009.
[12] A.V. Gnedin, Sequential selection of an increasing subsequence from a sample of random size, J. Appl. Probab. 36 (4) (1999) 1074–1085.
[13] P. Hall, C.C. Heyde, Martingale Limit Theory and its Application, Academic Press, Inc., New York, London, 1980.
[14] J.M. Hammersley, A few seedlings of research, in: Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. I: Theory of Statistics, Univ. California Press, Berkeley, CA, 1972, pp. 345–394.
[15] P. Jacquet, W. Szpankowski, Analytical de-Poissonization and its applications, Theoret. Comput. Sci. 201 (1–2) (1998) 1–62.
[16] J.H. Kim, On increasing subsequences of random permutations, J. Combin. Theory Ser. A 76 (1) (1996) 148–155.
[17] J.F.C. Kingman, Subadditive ergodic theory, Ann. Probab. 1 (1973) 883–909.
[18] J. Korevaar, Tauberian Theory: A Century of Developments, in: Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 329, Springer-Verlag, Berlin, 2004.
[19] B.F. Logan, L.A. Shepp, A variational problem for random Young tableaux, Adv. Math. 26 (2) (1977) 206–222.
[20] D.L. McLeish, Dependent central limit theorems and invariance principles, Ann. Probab. 2 (1974) 620–628.
[21] A.M. Odlyzko, E.M. Rains, On longest increasing subsequences in random permutations, in: Analysis, Geometry, Number Theory: The Mathematics of Leon Ehrenpreis (Philadelphia, PA, 1998), in: Contemp. Math., vol. 251, Amer. Math. Soc., Providence, RI, 2000, pp. 439–451.
[22] S. Pilpel, Descending subsequences of random permutations, J. Combin. Theory Ser. A 53 (1) (1990) 96–116.
[23] W. Rhee, M. Talagrand, A note on the selection of random variables under a sum constraint, J. Appl. Probab. 28 (4) (1991) 919–923.
[24] D. Romik, The Surprising Mathematics of Longest Increasing Subsequences, Cambridge University Press, Cambridge, 2015.

[25] S.M. Samuels, J.M. Steele, Optimal sequential selection of a monotone sequence from a random sample, Ann. Probab. 9 (6) (1981) 937–947.
[26] T. Tao, Poincaré's Legacies, Pages from Year Two of a Mathematical Blog. Part II, American Mathematical Society, Providence, RI, 2009.
[27] S.M. Ulam, Monte Carlo calculations in problems of mathematical physics, in: Modern Mathematics for the Engineer: Second Series, McGraw-Hill, New York, 1961, pp. 261–281.
[28] A.M. Veršik, S.V. Kerov, Asymptotic behavior of the Plancherel measure of the symmetric group and the limit form of Young tableaux, Dokl. Akad. Nauk SSSR 233 (6) (1977) 1024–1027.