ITÔ CALCULUS

J. MICHAEL STEELE

ABSTRACT. This entry for the *Encyclopedia of Actuarial Sciences* provides an introduction to the Itô Calculus that emphasizes the definition of the Itô integral and the description of Itô's Formula, the most widely used result in the Itô Calculus.

KEY WORDS: Itô Calculus, Itô's Formula, stochastic integrals, martingale, Brownian motion, diffusion process, Box calculus, harmonic function.

1. First Contact with Itô Calculus

From the practitioner's point of view, the Itô calculus is a tool for manipulating those stochastic processes which are most closely related to Brownian motion. The central result of the theory is the famous Itô formula which one can write in evocative shorthand as

$$df(t, B_t) = f_t(t, B_t)dt + \frac{1}{2} f_{xx}(t, B_t)dt + f_x(t, B_t)dB_t.$$

Here the subscripts denote partial derivatives, and the differential dt has the same interpretation that it has in ordinary calculus. On the other hand, the differential dB_t is a new object that needs some care to be properly understood.

At some level, one can think of dB_t as "increment of Brownian motion," but, even allowing this, one must somehow stir into those thoughts the considerations that would make dB_t independent of the information which one gains from observing $\{B_s : 0 \le s \le t\}$, the path of Brownian motion up to time t. The passive consumer can rely on heuristics such as these to follow some arguments of others, but an informal discussion of dB_t cannot take one very far.

To gain an honest understanding of the Itô integral one must spend some time with its formal definition. The time spent need not be burdensome, and one can advisably gloss over some details on first reading. Nevertheless, without some exposure to its formal definition, the Itô integral can only serve as a metaphor.

2. Itô Integration in Context

The Itô integral is a mathematical object which is only roughly analogous to the traditional integral of Newton and Leibniz. The real driving force behind the definition — and the effectiveness — of the Itô integral is that it carries the notion of a martingale transform from discrete time into continuous time.

The resulting construction is important for several reasons. First, it provides a systematic method for building new martingales, but it also provides the modeler with new tools for specifying stochastic processes in terms of "differentials." Initially these specifications are primarily symbolic, but typically they can be given rigorous interpretations which in turn alow one to forge a systematic connection between stochastic processes and the classical fields of ordinary and partial differential equations. The resulting calculus for stochastic processes turns out to be exceptionally fruitful both in theory and in practice, and the Itô calculus is now widely regarded as one of the most successful innovations of modern probability theory.

The aims addressed here are necessarily limited since even a proper definition of the Itô integral can take a dozen pages. Nevertheless, after a brief introduction to Itô calculus it is possible to provide (i) a sketch of the formal definition of the Itô integral, (ii) a summary of the key features of the integral, and (iii) some discussion of the widely used *Itô Formula.* For a more complete treatment of these topics, as well as more on their connections to issues of importance for actuarial science, one can consult Baxter and Rennie [1] or Steele [4]. For additional perspective on the theory of stochastic calculus one can consult Karatzas and Shreve [2] or Protter [3].

3. The Itô Integral: A Three Step Definition

If $\{B_t : 0 \leq t \leq T\}$ denotes Brownian motion on the finite interval $[0, T]$ and if $\{f(\omega, t) : 0 \leq t \leq T\}$ denotes a well-behaved stochastic process whose specific qualifications will be given later, then the Itô integral is a random variable which is commonly denoted by

$$I(f)(\omega) = \int_0^T f(\omega, t)\, dB_t. \tag{1}$$

This notation is actually somewhat misleading since it tacitly suggests that the integral on the right may be interpreted in a way that is analogous to the classical Riemann-Stieltjes integral. Unfortunately, such an interpretation is not possible on an ω-by-ω basis because almost all paths of Brownian motion fail to have bounded variation.

The definition of the Itô integral requires a more subtle limit process, which perhaps is best viewed as having three steps. In the first of these one simply isolates a class of simple integrands where one can say that the proper definition of the integral is genuinely obvious. The second step calls on a continuity argument which permits one to extend the definition of the integral to a larger class of natural processes. In the third step one then argues that there exists a continuous martingale which tracks the value of the Itô integral when it is viewed as a function of its upper limit; this martingale provides us with a view the Itô integral as a process.

The First Step: Definition on $\mathcal{H}_0^2$

The integrand of an Itô integral must satisfy some natural constraints, and, to detail these, we first let $\mathcal{B}$ denote the smallest σ-field that contains all of the open subsets of $[0,T]$; that is, we let $\mathcal{B}$ denote the set of *Borel* sets of $[0,T]$. We then take $\{\mathcal{F}_t\}$ to be the standard Brownian filtration, and for each $t \geq 0$ we take $\mathcal{F}_t \times \mathcal{B}$ to be the smallest σ-field that contains all of the product sets $A \times B$ where $A \in \mathcal{F}_t$ and $B \in \mathcal{B}$. Finally, we say $f(\cdot,\cdot)$ is *measurable* provided that $f(\cdot,\cdot)$ is $\mathcal{F}_T \times \mathcal{B}$ measurable, and we will say that $f(\cdot,\cdot)$ is *adapted* provided that $f(\cdot,t)$ is $\mathcal{F}_t$ measurable for each $t \in [0,T]$. One then considers the class $\mathcal{H}^2 = \mathcal{H}^2[0,T]$ of all measurable adapted functions f that are square-integrable in the sense that

$$E\left[\int_0^T f^2(\omega,t)\,dt\right] = \int_\Omega \int_0^T f^2(\omega,t)\,dtdP(\omega) < \infty. \tag{2}$$

If we write $L^2(dt \times dP)$ to denote the set of functions that satisfy (2), then by definition we have $\mathcal{H}^2 \subset L^2(dt \times dP)$. In fact $\mathcal{H}^2$ turns out to be one of the most natural domains for the definition and application of the Itô integral.

If we take $f(\omega,t)$ to be the indicator of the interval $(a,b] \subset [0,T]$, then $f(\omega,t)$ is trivially an element of $\mathcal{H}^2$, and in this case we quite reasonably want to define the Itô integral by the relation

$$I(f)(\omega) = \int_a^b dB_t = B_b - B_a. \tag{3}$$

Also, since one wants the Itô integral to be linear, the identity (3) will determine how $I(f)$ must be defined for a relatively large class of integrands. Specifically, if we let $\mathcal{H}_0^2$ denote the subset of $\mathcal{H}^2$ that consists of all functions that may be written as a finite sum of the form

$$f(\omega,t) = \sum_{i=0}^{n-1} a_i(\omega) 1(t_i < t \leq t_{i+1}), \tag{4}$$

where $a_i \in \mathcal{F}_{t_i}$, $E(a_i^2) < \infty$, and $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, then linearity and the equation (3) determine the value of I on $\mathcal{H}_0^2$. Now, for functions of the form (4) one simply defines $I(f)$ by the identity

$$I(f)(\omega) = \sum_{i=0}^{n-1} a_i(\omega)\{B_{t_{i+1}} - B_{t_i}\}. \tag{5}$$

This formula completes the first step, the definition of I on $\mathcal{H}_0^2$, though naturally one must check that this definition is unambiguous; that is, one must show that if f has two representations of the form (4) then the sums given by (5) provide the same values for $I(f)$.

The Second Step: Extension to $\mathcal{H}^2$

We now need to extend the domain of I from $\mathcal{H}_0^2$ to all of $\mathcal{H}^2$, and the key is to first show that $I : \mathcal{H}_0^2 \to L^2(dP)$ is an appropriately continuous mapping. In fact the following fundamental lemma tells us much more.

Lemma 1 (Itô's Isometry on $\mathcal{H}_0^2$). *For $f \in \mathcal{H}_0^2$ we have*

$$||I(f)||_{L^2(dP)} = ||f||_{L^2(dP\times dt)}. \tag{6}$$

By the linearity of $I : \mathcal{H}_0^2 \to L^2(dP)$, the Itô's Isometry Lemma implies that I takes equidistant points in $\mathcal{H}_0^2$ to equidistant points in $L^2(dP)$, so, in particular, I maps a Cauchy sequence in $\mathcal{H}_0^2$ into a Cauchy sequence in $L^2(dP)$. The importance of this observation is underscored by the next lemma which asserts that any $f \in \mathcal{H}^2$ can be approximated arbitrarily well by elements of $\mathcal{H}_0^2$.

Lemma 2 ($\mathcal{H}_0^2$ is Dense in $\mathcal{H}^2$). *For any $f \in \mathcal{H}^2$, there exists a sequence $\{f_n\}$ with $f_n \in \mathcal{H}_0^2$ such that*

$$||f - f_n||_{L^2(dP\times dt)} \to 0 \text{ as } n \to \infty.$$

Now, for any $f \in \mathcal{H}^2$, this approximation lemma tells us that there is a sequence $\{f_n\} \subset \mathcal{H}_0^2$ such that f_n converges to f in $L^2(dP \times dt)$. Also, for each n the integral $I(f_n)$ is given explicitly by formula (5), so the obvious idea is to define $I(f)$ as the limit of the sequence $I(f_n)$ in $L^2(dP)$; that is, we set

$$I(f) \stackrel{\text{def}}{=} \lim_{n\to\infty} I(f_n), \tag{7}$$

where the detailed interpretation of equation (7) is that the random variable $I(f)$ is the unique element of $L^2(dP)$ such that $||I(f_n) - I(f)||_{L^2(dP)} \to 0$ as $n \to \infty$. This completes the definition of $I(f)$, except for one easy exercise; it is still necessary to check that the random variable $I(f)$ does not depend on the specific choice that one makes for the approximating sequence $\{f_n : n = 1, 2, ...\}$.

The Third Step: Itô's Integral as a Process

The map $I\colon \mathcal{H}^2 \mapsto L^2(dP)$ permits one to define the Itô integral over the interval $[0, T]$, but to connect the Itô integral with stochastic processes we need to define the Itô's integral on $[0, t]$ for each $0 \le t \le T$ so that when viewed collectively these integrals will provide a continuous stochastic process.

This is the most delicate step in the construction of the Itô integral, but it begins with a straightforward idea. If one sets

$$m_t(\omega, s) = \begin{cases} 1 & \text{if } s \in [0, t] \\ 0 & \text{otherwise,} \end{cases}$$

then for each $f \in \mathcal{H}^2[0, T]$ the product $m_t f$ is in $\mathcal{H}^2[0, T]$, and $I(m_t f)$ is a well-defined element of $L^2(dP)$. A natural candidate for the process version of Itô's integral is then given by

$$X'_t(\omega) = I(m_t f)(\omega). \tag{8}$$

Sadly, this candidate has problems; for each $0 \le t \le T$ the integral $I(m_t f)$ is only defined as an element of $L^2(dP)$, so the value of $I(m_t f)$ can be specified *arbitrarily* on any set $A_t \in \mathcal{F}_t$ with $P(A_t) = 0$. The union of the A_t over all t in $[0, T]$ can be as large as the full set Ω, so, in the end, the process X'_t suggested by (8) might not be continuous for any $\omega \in \Omega$.

This observation is troubling, but it is not devastating. With care (and help from Doob's maximal inequality) one can prove that there exists a unique continuous martingale X_t which agrees with X'_t with probability one for each fixed $t \in [0, T]$. The next theorem gives a more precise statement of this crucial fact.

Theorem 1 (Itô Integrals as Martingales)**.** *For any $f \in \mathcal{H}^2[0, T]$, there is a process $\{X_t : t \in [0, T]\}$ that is a continuous martingale with respect to the standard Brownian filtration $\mathcal{F}_t$ and such that the event*

$$\{\, \omega : X_t(\omega) = I\,(\, m_t f\,)\,(\omega)\,\} \tag{9}$$

has probability one for each $t \in [0, T]$.

This theorem now completes the definition of the Itô integral of $f \in \mathcal{H}^2$; specifically, the process $\{X_t : 0 \le t \le T\}$ is a well-defined continuous martingale, and the Itô integral of f is *defined* by the relation

$$\int_0^t f(\omega, t)\, dB_t \stackrel{\text{def}}{=} X_t \quad \text{for} \quad t \in [0, T]. \tag{10}$$

An Extra Step: Itô's Integral on $\mathcal{L}^2_{\text{LOC}}$

The class $\mathcal{H}^2$ provides one natural domain for the Itô integral, but with a little more work one can extend the Itô integral to a larger space which one can argue is the *most* natural domain for the Itô integral. This space is known as $\mathcal{L}^2_{\text{LOC}}$, and it consists of all adapted, measurable functions $f : \Omega \times [0, T] \mapsto \mathbb{R}$ for which we have

$$P\left(\int_0^T f^2(\omega, t)\, dt < \infty \right) = 1. \tag{11}$$

Naturally $\mathcal{L}^2_{\text{LOC}}$ contains $\mathcal{H}^2$, but $\mathcal{L}^2_{\text{LOC}}$ has some important advantages over $\mathcal{H}^2$. In particular, for any continuous $g : \mathbb{R} \to \mathbb{R}$, the function given by $f(\omega, t) = g(B_t)$ is in $\mathcal{L}^2_{\text{LOC}}$ simply because for each ω the continuity of Brownian motion implies that the mapping $t \mapsto g(B_t(\omega))$ is bounded on $[0, T]$.

To indicate how the Itô integral is extended from $\mathcal{H}^2_0$ to $\mathcal{L}^2_{\text{LOC}}$, we first note that an increasing sequence of stopping times is called an $\mathcal{H}^2[0, T]$ *localizing sequence* for f provided that one has

$$f_n(\omega, t) = f(\omega, t) 1(t \le \nu_n) \in \mathcal{H}^2[0, T] \text{ for all } n \tag{12}$$

and provided that one has

$$P\left(\bigcup_{n=1}^{\infty}\{\,\omega\colon \nu_n = T\,\}\right) = 1. \tag{13}$$

For example, one can easily check that if $f \in \mathcal{L}^2_{\mathrm{LOC}}[0,T]$ then the sequence

$$\tau_n = \inf\left\{s\colon \int_0^s f^2(\omega,t)\,dt \geq n \;\text{ or }\; s \geq T\right\} \tag{14}$$

is an $\mathcal{H}^2[0,T]$ localizing sequence for f.

Now, to see how the Itô integral is defined on $\mathcal{L}^2_{\mathrm{LOC}}$, we take $f \in \mathcal{L}^2_{\mathrm{LOC}}$ and let $\{\nu_n\}$ be a localizing sequence for f; for example, one could take $\nu_n = \tau_n$ where τ_n is defined by (14). Next, for each n, take $\{X_{t,n}\}$ to be the unique continuous martingale on $[0,T]$ that is a version of the Itô integral of $I(m_t g)$ and where $g(\omega,s) = f(\omega,s)1(s \leq \nu_n(\omega))$. Finally, we define the Itô integral for $f \in \mathcal{L}^2_{\mathrm{LOC}}[0,T]$ to be the process given by the limit of the processes $\{X_{t,n}\}$ as $n \to \infty$. More precisely, one needs to show that there is a unique continuous process $\{X_t\colon 0 \leq t \leq T\}$ such that

$$P\left(X_t = \lim_{n\to\infty} X_{t,n}\right) = 1 \text{ for all } t \in [0,T]; \tag{15}$$

so, in the end, we can take the process $\{X_t\}$ to be our Itô integral of $f \in \mathcal{L}^2_{\mathrm{LOC}}$ over $[0,t]$, $0 \leq t \leq T$. In symbols, we *define* the Itô integral of f by setting

$$\int_0^t f(\omega,s)\,dB_s \stackrel{\text{def}}{=} X_t(\omega) \;\text{ for } t \in [0,T]. \tag{16}$$

Some work is required to justify this definition, and in particular one needs to show that he defining limit (16) does not depend on the choice that we make for the localizing sequence, but once these checks are made, the definition of the Itô integral on $\mathcal{L}^2_{\mathrm{LOC}}$ is complete. The extension of the Itô integral from $\mathcal{H}^2$ to $\mathcal{L}^2_{\mathrm{LOC}}$ introduces some intellectual overhead, and one may wonder if the light is worth the candle: be assured, it is. Because of the extension of the Itô integral from $\mathcal{H}^2$ to $\mathcal{L}^2_{\mathrm{LOC}}$ we can now consider the Itô integral of any continuous function of Brownian motion. Without the extension, this simple and critically important case would have been out of our reach.

Some Perspective and Two Intuitive Representations

In comparison with the traditional integrals, one may find that the time and energy required to define the Itô integral is substantial. Even here, where all proofs and verifications have been omitted, one needed several pages to make the definition explicit. Yet, oddly enough, one of the hardest parts of the theory of Itô calculus is the definition of the integral; once the definition is complete, the calculations which one finds in the rest of the theory are largely in line with the familiar calculations of analysis.

The next two propositions illustrate this phenomenon, and they also add to our intuition about the Itô integral since they reassure us that at least in some special cases the Itô integrals can be obtained by formulas that remind us of the traditional Riemann limits. Nevertheless, one must keep in mind that even though these formulas are intuitive, they covertly lean on all the labor that is required for formal definition of the Itô integral. In fact, without that labor, the assertions would not even be well-defined.

Proposition 1 (Riemann Representation)**.** *For any continuous* $f : \mathbb{R} \to \mathbb{R}$*, if we take the partition of* $[0, T]$ *given by* $t_i = iT/n$ *for* $0 \leq i \leq n$*, then we have*

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(B_{t_{i-1}})(B_{t_i} - B_{t_{i-1}}) = \int_0^T f(B_s)\, dB_s, \tag{17}$$

where the limit is understood in the sense of convergence in probability.

Proposition 2 (Gaussian Integrals)**.** *If* $f \in C[0, T]$*, then the process defined by*

$$X_t = \int_0^t f(s)\, dB_s \quad t \in [0, T] \tag{18}$$

is a mean zero Gaussian process with independent increments and with covariance function

$$\mathrm{Cov}(X_s, X_t) = \int_0^{s\wedge t} f^2(u)\, du. \tag{19}$$

Moreover, if we take the partition of $[0, T]$ *given by* $t_i = iT/n$ *for* $0 \leq i \leq n$ *and* t_i^* *satisfies* $t_{i-1} \leq t_i^* \leq t_i$ *for all* $1 \leq i \leq n$*, then we have*

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(t_i^*)(B_{t_i} - B_{t_{i-1}}) = \int_0^T f(s)\, dB_s,$$

where the limit is understood in the sense of convergence in probability.

4. Itô's Formula

The most important result in the Itô calculus is Itô's formula, for which there are many different versions. We will first consider the simplest.

Theorem 2 (Itô's Formula)**.** *If the function* $f : \mathbb{R} \mapsto \mathbb{R}$ *has a continuous second derivative, then one has the representation*

$$f(B_t) = f(0) + \int_0^t f'(B_s)\, dB_s + \frac{1}{2}\int_0^t f''(B_s)\, ds. \tag{20}$$

There are several interpretations of this formula, but perhaps it is best understood as a version of the fundamental theorem of calculus. In one way the analogy is apt; this formula can be used to calculate Itô integrals in much the same way that the fundamental theorem of calculus can be used to calculate traditional definite integrals. In other ways the analogy is less apt; for example, one has an extra term in the right hand sum, and, more important, the expression B_s that appears in the first integral is completely unlike the dummy variable which it would represent if this integral were understood in the sense of Riemann.

A Typical Application

If $F \in C^2(\mathbb{R})$ and $F' = f$ with $F(0) = 0$, then Itô's formula can be rewritten as

$$\int_0^t f(B_s)\, dB_s = F(B_t) - \frac{1}{2}\int_0^t f'(B_s)\, ds, \tag{21}$$

and in this form it is evident that Itô's formula can be used to calculate many interesting Itô integrals. For example, if we take $F(x) = x^2/2$ then $f(B_s) = B_s$, $f'(B_s) = 1$, and we find

$$\int_0^t B_s\, dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t. \tag{22}$$

In other words, the Itô integral of B_s on $[0, t]$ turns out to be just a simple function of Brownian motion and time. Moreover, we know that this Itô integral is a martingale, so, among other things, this formula reminds us that $B_t^2 - t$ is a martingale, a basic fact that can be checked several ways.

A second way to interpret Itô's formula is as a decomposition of $f(B_t)$ into components that are representative of noise and signal. The first integral of equation (20) has mean zero and it captures information about the local variability of $f(B_t)$ while the second integral turns out to capture all of the information about the drift of $f(B_t)$. In this example we see that B_t^2 can be understood as a process with a "signal" equal to t and a "noise component" N_t that is given by the Itô integral

$$N_t = 2\int_0^t B_s\, dB_s.$$

Brownian Motion and Time

The basic formula (20) has many useful consequences, but its full effect is only realized when it is extended to accommodate function of Brownian motion *and time*.

Theorem 3 (Itô's Formula with Space and Time Variables)**.** *If a function $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ has a continuous derivative in its first variable and a continuous second derivative in its second variable, then one has the representation*

$$f(t, B_t) = f(0,0) + \int_0^t \frac{\partial f}{\partial x}(s, B_s)\, dB_s + \int_0^t \frac{\partial f}{\partial t}(s, B_s)\, ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)\, ds.$$

One of the most immediate benefits of this version of Itô's formula is that gives one a way to recognize when $f(t, B_t)$ is a local martingale. Specifically, if $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and if $f(t, x)$ satisfies the equation

$$\frac{\partial f}{\partial t} = -\frac{1}{2}\frac{\partial^2 f}{\partial x^2}, \tag{23}$$

then the space-time version of Itô's formula immediately tells us that X_t can be written as the Itô integral of $f_x(t, B_t)$. Such an integral is always a local martingale, and if the representing integrand is well-behaved in the sense that

$$E\left[\int_0^T \left\{\frac{\partial f}{\partial x}\right\}^2 (t, B_t)\, dt\right] < \infty, \tag{24}$$

then in fact X_t is an honest martingale on $0 \le t \le T$.

To see the ease with which this criterion can be applied, consider the process $M_t = \exp(\alpha B_t - \alpha^2 t/2)$ corresponding to $f(x, t) = \exp(\alpha x - \alpha^2 t/2)$. In this case we have

$$\frac{\partial f}{\partial t} = -\frac{1}{2}\alpha^2 f \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2} = \alpha^2 f,$$

so the differential condition (23) is satisfied. As a consequence we see that M_t is a local martingale, but it is also clear that M_t is an honest martingale since the $\mathcal{H}^2$ condition (24) is immediate. The same method can be used to show that $M_t = B_t^2 - t$ and $M_t = B_t$ are martingales. One only has to note that $f(t, x) = x^2 - t$ and $f(t, x) = x$ satisfy the PDE condition (23), and in both cases we have $f(t, B_t) \in \mathcal{H}^2$.

Finally, we should note that there is a perfectly analogous vector version of Itô's formula, and it provides us with a corresponding criterion for a function of time and several Brownian motions to be a local martingale.

Theorem 4 (Itô's Formula —Vector Version)**.** *If $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ and if $\vec{B}_t$ is standard Brownian motion in $\mathbb{R}^d$, then*

$$df(t, \vec{B}_t) = f_t(t, \vec{B}_t)\, dt + \nabla f(t, \vec{B}_t) \cdot d\vec{B}_t + \frac{1}{2}\Delta f(t, \vec{B}_t)\, dt.$$

From this formula we see that if If $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$ and $\vec{B}_t$ is standard Brownian motion in $\mathbb{R}^d$, then the process $M_t = f(t, \vec{B}_t)$ is a local martingale provided that

$$f_t(t, \vec{x}) = -\frac{1}{2}\Delta f(t, \vec{x}).$$

If we specialize this observation to functions that depend only on $\vec{x}$, we see that the process $M_t = f(\vec{B}_t)$ is a local martingale provided that $\Delta f = 0$; that is, $M_t = f(\vec{B}_t)$ is a local martingale provide that f is a harmonic function. This observation provides a remarkably fecund connection between Brownian motion and classical potential theory, which is one of the richest branches of mathematics.

5. The Itô Shorthand and More General Integrals

Formulas of the Itô calculus can be lengthy when written out in detail, so it is natural that shorthand notation has been introduced. In particular, if X_t is a process that with a representation of the form

$$X_t = \int_0^t a(\omega, s)\, ds + \int_0^t b(\omega, s)\, dB_s, \tag{25}$$

for some suitable processes $a(\omega, s)$ and $b(\omega, s)$, then it is natural to write this relationship more succinctly with the shorthand

$$dX_t = a(\omega, t)\, dt + b(\omega, t)\, dB_t, \quad X_0 = 0. \tag{26}$$

Expressions such as dX_t, dB_t, and dt are highly evocative, and the intuition one forms about them is important for the effective use of the Itô calculus. Nevertheless, in the final analysis, one must always keep in mind that entities like dX_t, dB_t, and dt draw all of their meaning from their longhand interpretation. To prove a result that relies on these freeze-dried expressions, one must be ready — at least in principle — to first reconstitute them as they appear the original expression (25).

With this caution in mind, one can still use the intuition provided by terms like dX_t to suggest new results, and, when one follows this path, it is natural to *define* the dX_t integral of $f(t, \omega)$ by setting

$$\int_0^t f(\omega, s)\, dX_s \stackrel{\text{def}}{=} \int_0^t f(\omega, s) a(\omega, s)\, ds + \int_0^t f(\omega, s) b(\omega, s)\, dB_s. \tag{27}$$

Here, of course, one must impose certain restrictions on $f(\omega, t)$ for the last two integrals make sense, but it would certainly suffice to assume that $f(\omega, t)$ is adapted and that it satisfies the integrability conditions:

- $f(\omega, s)\, a(\omega, s) \in L^1(dt)$ for all ω in a set of probability one and
- $f(\omega, s) b(\omega, s) \in \mathcal{L}^2_{\text{LOC}}$.

6. From Itô's Formula to the Box Calculus

The experience with Itô's formula as a tool for understanding the dB_t integrals now leaves one with a natural question: Is there an appropriate analog of Itô's formula for dX_t integrals? That is, if the process X_t can be written as a stochastic integral of the form (27) and if $g(t, y)$ is a smooth function, can we then write the process $Y_t = g(t, X_t)$ as a sum of terms which includes a dX_t integral?

Naturally there is an affirmative answer to this question, and it turns out to be nicely express with help from a simple formalism that is usually called the *box calculus*, though the term *box algebra* would be more precise. This is an algebra for the set $\mathcal{A}$ of linear combinations of the formal symbols dt and dB_t where adapted functions are regarded as the scalars. In this algebra, the addition operation is just the usual algebraic addition, and products are then computed by the traditional rules of associativity and transitivity together with a multiplication table for the special symbols dt and dB_t. The new rules one uses are simply

$$dt \cdot dt = 0, \quad dt \cdot dB_t = 0, \quad \text{and} \quad dB_t \cdot dB_t = dt.$$

As an example of the application of these rule are applied, one can check that the product

$$(a\,dt + b\,dB_t) \cdot (\alpha\,dt + \beta\,dB_t)$$

can be simplified by associativity and commutativity to give

$$a\alpha\,dt \cdot dt + a\beta\,dt \cdot dB_t + b\alpha\,dB_t \cdot\, dt + b\beta\,dB_t \cdot dB_t = b\beta\,dt.$$

If one uses this formal algebra for the process X_t which we specified in longhand by (25) or in shorthand by (26), then one has the following *general version of Itô's formula*:

$$df(t, X_t) = f_t(t, X_t)\,dt + f_x(t, X_t)\,dX_t + \frac{1}{2} f_{xx}(t, X_t)\,dX_t \cdot dX_t. \tag{28}$$

This simple formula is exceptionally productive formula, and it summarizes a vast amount of useful information.

In the simplest case, we see by setting $X_t = B_t$ that the formula (28) quietly recaptures space-time version of Itô's formula. Still, it is easy to go much farther. For example, if we take $X_t = \mu t + \sigma B_t$ so X_t is Brownian motion with with drift, or if we take $X_t = \exp(\mu t + \sigma B_t)$ so X_t is Geometric Brownian motion, the general Itô formula (28) painlessly produces formulas for $df(t, X_t)$ which other wise could be won only by applying the space-time version of Itô's formula together with tedious and error prone applications of the chain rule.

To address more novel examples, one naturally needs to provide a direct proof of the general Itô formula, a proof that does not go through the space-time version of Itô's formula. Fortunately, such a proof is not difficult, and it is not even necessary to introduce any particularly new technique. In essence, a properly modified repetition of the proof of the space-time Itô's formula will suffice.

7. Concluding Perspectives

Itô's calculus provides the users of stochastic models with a theory that maps forcefully into some of the most extensively developed area of mathematics, including the theory of ordinary differential equations, partial differential equations, and the theory of harmonic functions. Itô's calculus has also led to more sophisticated versions of stochastic integration where the role of Brownian motion can be replaced by any Lévy process, or even by more general processes. Moreover, Itô calculus has had a central role in some of the most important developments of financial theory, including the Merton and Black-Scholes theories of option pricing.

Too be sure, there is some overhead involved in the acquisition of a fully functioning background in the Itô calculus. One also faces substantial limitations on the variety of models that are supported by the Itô calculus;

the ability of diffusion models to capture the essence of empirical reality can be marvellous, but in some contexts the imperfections are all too clear. Still, despite its costs and its limitations, the Itô calculus stands firm as one of the most effective tools we have for dealing with models which hope to capture the realities of randomness and risk.

References

Baxter, M. and Rennie, A. (1996). *Financial Calculus: An Introduction to Derivative Pricing*, Cambridge University Press, Cambridge.

Karatzas, I. and Shreve, S. (1991). *Brownian Motion and Stochastic Calculus*, 2nd Ed., Springer-Verlag, New York.

Protter, P. (1995). *Stochastic Integration and Differential Equations: A New Approach*, Springer-Verlag, New York.

Steele, J.M. (2001). *Stochastic Calculus and Financial Applications*, Springer-Verlag, New York.

Department of Statistics, Wharton School, University of Pennsylvania, Huntsman Hall 4000, University of Pennsylvania, Philadelphia PA 19104

E-mail address: steele@wharton.upenn.edu