GIRSANOV'S THEOREM : A CLASS NOTE EXPLOITING REAL ANALYTIC CONTINUATION

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Abstract. This classroom note (not for publication) proves Girsanov's Theorem by a special kind of real-variable analytic continuation argument. While this interesting plan (due to R.S. Liptser and A.N. Shiryayev) is no longer the most efficient path Girsanov's theorem, it is still instructive. Moreover, the argument is likely to find many other applications. The Liptser-Shiryayev argument was used in the first edition of *Stochastic Calculus and Financial Applications*, but in the second edition edition, it was replaced by a quite different argument of N. V. Krylov (2002).

1. Exponential Martingales and Novikov's Condition

One of the key issues in the use of Girsanov theory is the articulation of circumstances under which an exponential local martingale is an honest martingale. Sometimes, we can be content with a simple sufficient condition such as boundedness, but at other times we need serious help. There are now several useful criteria, but the most well-known is surely the 1972 criterion of A.A. Novikov.

Theorem 1 (The Novikov Sufficient Condition)**.**
For every $\mu \in \mathcal{L}^2_{\mathrm{LOC}}[0,T]$, the process defined by

$$M_t(\mu) = \exp\left(\int_0^t \mu(\omega,s)\,dB_s - \frac{1}{2}\int_0^t \mu^2(\omega,s)\,ds\right) \tag{1}$$

is a martingale, provided that μ satisfies the Novikov condition

$$E\left[\exp\left(\frac{1}{2}\int_0^T \mu^2(\omega,s)\,ds\right)\right] < \infty. \tag{2}$$

2. Understanding the Condition

One of Pólya's bits of advice in *How to Solve It* is to "understand the condition." Like many of the other pieces of Pólya's problem-solving advice, this seems like such basic common sense that we may not take the suggestion as seriously as perhaps we should. Here the suggestion is particularly wise.

When we look at the condition (2) and angle for a deeper understanding, one of the observations that may occur to us is that if μ satisfies the condition then so does $\lambda\mu$ for any $|\lambda| \leq 1$. At first, there may not seem like there is much force to this added flexibility, but it offers the seed of a good plan.

Date: December 1, 2006.
1991 *Mathematics Subject Classification.* Primary: 60D05; Secondary: 60C05,68U05,62G99.
Key words and phrases. Girsanov's theorem, exponential martingales, real analytic continuation.

3. A Plan Suggested by Power Series

From Itô's formula we find that $M_t(\mu)$ is an local martingale. Since it is non-negative, it Fatou's lemma then implies that $M_t(\mu)$ is a supermartingale. We also know that a supermartigale will be an honest martingale on $[0,T]$ if we can show $E[M_T(\mu)] = EM_0(\mu)$. This modest observation suggests a marvelous plan.

If we introduce the function $H(\lambda) = E[M_T(\lambda\mu)]$, where λ is a real parameter, then the proof is complete if we show $H(1) = 1$, but, if the theorem is true (as we strongly suspect!), we should actually have $H(\lambda) = 1$ for all $|\lambda| \le 1$. It is trivial that $H(0) = 1$, and from the definition of M_t we might suspect that we would have an easier time proving $H(\lambda) = 1$ for $\lambda \in (-1,0]$ than for positive $\lambda > 0$. At this point, some experience with power series suggests that if we can prove $H(\lambda) = 1$ for all λ in an interval such as $(-1,0]$, then we should have great prospects of proving that $H(\lambda) = 1$ for all $\lambda \le 1$. Even without such experience, the plan should be at least modestly plausible, and, in any event, we will need to make some small modifications along the way.

4. First a Localization

As usual when working with local processes, we do well to slip in a localization that makes our life as easy as possible. Here, we want to study $M_t(\lambda\mu)$ for negative λ, so we want to make sure that the exponent in $M(\mu)$ is not too small. For this purpose, we will use the related process

$$Y_t = \int_0^t \mu(\omega,s)\,dB_s - \int_0^t \mu^2(\omega,s)\,ds$$

and introduce the stopping time

$$\tau_a = \inf\{\,t : Y_t = -a \text{ or } t \ge T\,\}.$$

The next proposition gives us some concrete evidence that our plan is on track.

Proposition 1. *For all $\lambda \le 0$, we have the identity*

$$E[M_{\tau_a}(\lambda\mu)] = 1. \tag{3}$$

Proof. As we have seen several times before, Itô's formula tells us that the process $dM_t(\lambda\mu)$ satisfies $dM_t(\lambda\mu) = \lambda\mu(\omega,s)M_t(\lambda\mu)\,dB_t$ and as a consequence we have the integral representation

$$M_{\tau_a}(\lambda\mu) = 1 + \int_0^{\tau_a} \lambda\mu(\omega,s)M_{\tau_a}(\lambda\mu)\,dB_s. \tag{4}$$

Now, to prove (3), we only need to show that the integrand in equation (4) is in $\mathcal{H}^2$, or, in other words, we must show

$$E\left[\int_0^{\tau_a} \mu^2(\omega,s)M^2_{\tau_a}(\lambda\mu)\,ds\right] < \infty. \tag{5}$$

Here we first note that for $s \le \tau_a$ we have

$$\begin{aligned} M_s(\lambda\mu) &= \exp\left(\lambda\int_0^s \mu(\omega,s)\,dB_s - \frac{\lambda^2}{2}\int_0^s \mu^2(\omega,s)\,ds\right) \\ &= \exp(\lambda Y_s)\exp\left((\lambda-\lambda^2/2)\int_0^s \mu^2(\omega,s)\,ds\right) \\ &\le \exp(a|\lambda|), \end{aligned} \tag{6}$$

where in the last step we use the definition of τ_a and the fact that $\lambda - \lambda^2/2 \leq 0$ for $\lambda \leq 0$. Next, we note that the simple bound $x^2 \leq 2\exp(x^2/2)$ and Novikov's condition combine to tell us that

$$E\left(\int_0^T \mu^2(\omega,s)\,ds\right) \leq 2E\left[\exp\left(\frac{1}{2}\int_0^T \mu^2(\omega,s)\,ds\right)\right] < \infty. \tag{7}$$

Finally, in view of the bounds (6) and (7), we see that equation (5) holds, so the proof of the proposition is complete. □

5. Power Series and Positive Coefficients

At this point, one might be tempted to expand $E(M_{\tau_a}(\lambda\mu))$ as a power series in λ in order to exploit the identity (3), but this frontal assault runs into technical problems. Fortunately, these problems can be avoided if we can manage to work with power series with nonnegative coefficients. The next lemma reminds us how pleasantly such series behave. To help anticipate how the lemma will be applied, we should note that the inequality (8) points toward the supermartingale property of $M_{\tau_a}(\lambda\mu)$ whereas the equality (9) connects with the identity that we just proved in Proposition 1.

Lemma 1. *If $\{c_k(\omega)\}$ is a sequence of nonnegative random variables and $\{a_k\}$ is a sequence of reals such that the two power series*

$$f(x,\omega) = \sum_{k=0}^{\infty} c_k(\omega)\,x^k \text{ and } g(x) = \sum_{k=0}^{\infty} a_k x^k$$

satisfy

$$E[f(x,\omega)] \leq g(x) < \infty \quad \text{for } x \in (-1,1] \tag{8}$$

and

$$E[f(x,\omega)] = g(x) \quad \text{for } x \in (-1,0], \tag{9}$$

then

$$E[f(1,\omega)] = g(1). \tag{10}$$

Proof. Since $c_k(\omega) \geq 0$, we can apply Fubini's theorem and the bound (8) to get

$$E[f(x,\omega)] = \sum_{k=0}^{\infty} E(c_k)\,x^k \leq g(x) < \infty \quad \text{for } x \in (-1,1], \tag{11}$$

so by the bound (8) we have

$$\sum_{k=0}^{\infty} E(c_k)\,x^k = \sum_{k=0}^{\infty} a_k x^k \quad \text{for all } x \in (-1,0]. \tag{12}$$

By the uniqueness of power series, the last identity tells us $E(c_k) = a_k$ for all $k \geq 0$, so we also have the identity (12) for all $x \in (-1,1)$. By the monotonicity of $f(x,\omega)$ on $[0,1)$, we can take the limit $x \uparrow 1$ in the identity $g(x) = E(f(x,\omega))$ on $[0,1)$ to conclude that $g(1) = E(f(1,\omega))$. □

6. Extending the Identity

By Proposition 1, we know that $E[M_{\tau_a}(\lambda\mu)] = 1$ for all $\lambda \le 0$, and we simply need to extend this identity to $\lambda \le 1$. When we write $M_t(\lambda\mu)$ in terms of Y_t, we find

$$M_t(\lambda\mu) = \exp\left(\lambda Y_t + (\lambda - \lambda^2/2)\int_0^t \mu^2(\omega, s)\, ds\right),$$

and the relationship of Y_t to the level a can be made more explicit if we consider

$$e^{\lambda a} M_t(\lambda\mu) = \exp\left(\lambda(Y_t + a) + (\lambda - \lambda^2/2)\int_0^t \mu^2(\omega, s)\, ds\right). \tag{13}$$

Now, if we reparameterize the preceding expression just a bit, we will be able to obtain a power series representation for $e^{\lambda a} M_{\tau_a}(\lambda\mu)$ with nonnegative coefficients.

Specifically, we first choose z so that $\lambda - \lambda^2/2 = z/2$, and we then solve the quadratic equation to find two candidates for λ. Only the root $\lambda = 1 - \sqrt{1-z}$ will satisfy $\lambda \le 1$ when $|z| \le 1$, so we will use the substitutions

$$\lambda - \lambda^2/2 = z/2 \text{ and } \lambda = 1 - \sqrt{1-z}$$

to replace the λ's by the z in the identity (13).

In these new variables, the power series for $e^{\lambda a} M_{\tau_a}(\lambda\mu)$ is given by

$$\begin{aligned} f(\omega, z) &= \exp\left((1 - \sqrt{1-z})(Y_{\tau_a} + a) + \frac{z}{2}\int_0^{\tau_a} \mu^2(\omega, s)\, ds\right) \\ &= \sum_{k=0}^{\infty} c_k(\omega) z^k, \end{aligned} \tag{14}$$

and, because the power series for e^z and $1 - \sqrt{1-z}$ have only positive coefficients, we see that $c_k(\omega) \ge 0$ for all $k \ge 0$.

Now, because $e^{(1-\sqrt{1-z})a} M_t((1 - \sqrt{1-z})\mu)$ is a supermartingale for any $z \le 1$, we can also take the expectation in equation (14) to find

$$E[f(\omega, z)] \le \exp(a(1 - \sqrt{1-z}) \stackrel{\text{def}}{=} g(z) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} a_k z^k. \tag{15}$$

The identity of Proposition 1 tells us that for all $z \in (-1, 0]$ we have

$$E[f(\omega, z)] = \exp(a(1 - \sqrt{1-z}) \stackrel{\text{def}}{=} g(z), \tag{16}$$

so all of the conditions of Lemma 1 are in place, and we can apply the lemma to conclude that

$$E[f(\omega, 1)] = e^a,$$

so when we unwrap the definition of f, we find

$$E[M_{\tau_a}(\mu)] = 1.$$

All that remains to complete the proof of Theorem 1 is to show that τ_a can be replaced by T in the previous identity.

7. Final Step: Delocalization

The natural plan is to let $a \to \infty$ in $E[M_{\tau_a}(\mu)] = 1$ so that we may conclude $E[M_T(\mu)] = 1$. This plan is easily followed. The first step is to note that the identity $E[M_{\tau_a}(\mu)] = 1$ gives us

$$\begin{aligned} 1 &= E[M_{\tau_a}(\mu)\mathbb{I}(\tau_a < T)] + E[M_{\tau_a}(\mu)\mathbb{I}(\tau_a = T)] \\ &= E[M_{\tau_a}(\mu)\mathbb{I}(\tau_a < T)] + E[M_T(\mu)\mathbb{I}(\tau_a = T)], \end{aligned}$$

and trivially we have

$$E[M_T(\mu)] = E[M_T(\mu)\mathbb{I}(\tau_a = T)] + E[M_T(\mu)\mathbb{I}(\tau_a < T)],$$

so we have

$$E[M_T(\mu)] = 1 - E[M_{\tau_a}(\mu)\mathbb{I}(\tau_a < T)] + E[M_T(\mu)\mathbb{I}(\tau_a < T)]. \tag{17}$$

Now, $Y_{\tau_a} = -a$ and

$$\begin{aligned} M_{\tau_a}(\mu)\mathbb{I}(\tau_a < T) &= \mathbb{I}(\tau_a < T) \exp\left(Y_{\tau_a} + \frac{1}{2}\int_0^{\tau_a} \mu^2(\omega, s)\, ds \right) \\ &\leq e^{-a} \exp\left(\frac{1}{2}\int_0^{T} \mu^2(\omega, s)\, ds \right), \end{aligned}$$

and the Novikov condition tells us the exponential has a finite expectation so as $a \to \infty$ we find

$$E[M_{\tau_a}(\mu)\mathbb{I}(\tau_a < T)] \leq e^{-a} E\left[\exp\left(\frac{1}{2}\int_0^{T} \mu^2(\omega, s)\, ds \right) \right] \to 0. \tag{18}$$

The continuity of Y_t implies that $\mathbb{I}(\tau_a < T) \to 0$ for all ω, and the supermartingale property gave us $E[M_T(\mu)] \leq 1$, so now by the dominated convergence theorem, we find

$$E[M_T(\mu)\mathbb{I}(\tau_a < T)] \to 0 \quad \text{as } a \to \infty. \tag{19}$$

Finally, if we apply the limit results (18) and (19) in the identity (17), then we see at last that $E[M_T(\mu)] = 1$ and we have confirmed that $\{M_t : 0 \leq t \leq T\}$ is an honest martingale.

8. Looking Back: The Nature of the Pattern

In our development of the martingale representation theorem we found an analogy between mathematical induction and the way we worked our way along a sequence of special case to the general theorem. The proof of Novikov's theorem followed a different pattern, but it also provides an analogy with induction.

We began with the trivial observation that $H(0) = 1$ which is analogous to the proposition $P(1)$ in mathematical induction. This observation motivated us to study the more general case $H(\lambda) = 1$ for $\lambda \leq 0$, and this is at least partially analogous to showing $P(n) \Rightarrow P(n+1)$. Finally, function theoretic facts were used to show that $H(\lambda) = 1$ for $\lambda \leq 1$. For this step the parallel is quite loose, but metaphorically at least one can view it as analogous to invoking the principle of mathematical induction.

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