Monotone subsequences in the sequence of fractional parts of multiples of an irrational*)

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1. Introduction

Hammersley [7] showed that if $X_1, X_2, \ldots$ is a sequence of independent identically distributed random variables whose common distribution is continuous, and if l_n^+ (l_n^-) denotes the length of the longest increasing (decreasing) subsequence of $X_1, X_2, \ldots, X_n$, then there is a constant c such that $\frac{l_n^+}{n^{\frac{1}{2}}} \to c$ and $\frac{l_n^-}{n^{\frac{1}{2}}} \to c$ in probability, as $n \to \infty$. Kesten [8] showed that in fact there is almost sure convergence. Logan and Shepp [11] proved that $c \geqq 2$, and recently Veršik and Kerov [13] have announced that $c=2$.

If α is an irrational then the sequence $\{\alpha\}, \{2\alpha\}, \ldots$ of fractional parts of multiples of α is uniformly distributed in the unit interval. Franklin [5] calls such a sequence a "Weyl sequence" and applies various tests to determine its quality as a pseudo-random sequence. In this spirit, it is reasonable to investigate $l_n^+(\alpha)$ and $l_n^-(\alpha)$, the lengths of the longest increasing and decreasing subsequences of $\{\alpha\}, \ldots, \{n\alpha\}$. We will be particularly interested in the behaviour of $\frac{l_n^+}{n^{\frac{1}{2}}}$ and $\frac{l_n^-}{n^{\frac{1}{2}}}$.

Some work along these lines was done by del Junco and Steele in [2]. Using discrepancy estimates, they were able to show that $\frac{\log l_n^+}{\log n} \to \frac{1}{2}$ and $\frac{\log l_n^-}{\log n} \to \frac{1}{2}$ for almost all α, and in particular for algebraic irrationals.

Here we shall be able to obtain more precise results by establishing the exact connection between l_n^+ (l_n^-) and the continued fraction expansion of α. We will find piecewise linear functions of n, λ_n^+ and λ_n^- whose vertices are explicitly determined by α, which satisfy $\lambda_n^+ - 2 < l_n^+ \leqq \lambda_n^+$ and $\lambda_n^- - 2 < l_n^- \leqq \lambda_n^-$. The results are precise enough to show, for example,

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that there is no α for which $\frac{l_n^+}{n^{\frac{1}{2}}}$ or $\frac{l_n^-}{n^{\frac{1}{2}}}$ tends to a limit. In fact $n^{\frac{1}{2}}$ is the correct order of magnitude of l_n^+ and l_n^- precisely when α has bounded partial quotients. For example, if $\alpha = \frac{5^{\frac{1}{2}}+1}{2}$, then

$$\liminf \frac{l_n^+}{n^{\frac{1}{2}}} = \frac{2}{5^{\frac{1}{4}}} = 1.337481\ldots$$

$$\limsup \frac{l_n^+}{n^{\frac{1}{2}}} = 5^{\frac{1}{4}} = 1.495349\ldots$$

This example gives the minimum value attainable for the difference $\limsup \frac{l_n^+}{n^{\frac{1}{2}}} - \liminf \frac{l_n^+}{n^{\frac{1}{2}}}$.

This last result invites comparison with a result of del Junco and Steele concerning the van der Corput sequence, to the effect that

$$\liminf \frac{l_n^+}{n^{\frac{1}{2}}} = 2^{\frac{1}{2}}$$

$$\limsup \frac{l_n^+}{n^{\frac{1}{2}}} = \frac{3}{2}.$$

An interesting contrast to the result for random sequences is that

$$(1) \qquad n \leqq l_n^+(\alpha)\, l_n^-(\alpha) \leqq 2n, \quad \text{for all } \alpha.$$

In fact $\limsup \frac{l_n^+ l_n^-}{n} = 2$ for all α, and $\liminf \frac{l_n^+ l_n^-}{n} = 1$ if α has unbounded partial quotients. The lower inequality is essentially the familiar result of Erdös and Szekeres [4], but the upper inequality is peculiar to the sequence $\{n\alpha\}$. Although it is an easy consequence of our formulas for λ_n^+ and λ_n^-, a more direct proof would be desirable. Note that, by contrast with (1), for random sequences $\frac{l_n^+ l_n^-}{n} \longrightarrow 4$.

The result (1), as well as aspects of the structure of the longest monotone subsequences, was suggested by a computation of l_n^+ and l_n^- for $\alpha = 2^{\frac{1}{2}}$, $\frac{5^{\frac{1}{2}}+1}{2}$ and e for $n \leqq 100{,}000$, using the algorithm of Fredman [6]. Only the values of n for which $l_n^+ > l_{n-1}^+$ or $l_n^- > l_{n-1}^-$ were printed, and these values suggested the connection with continued fractions.

The pattern of the proof is as follows. We first show that l_n^+ is the solution to a certain integer programming problem. We then define λ_n^+ to be the solution of the corresponding linear programming problem, so that obviously $\lambda_n^+ \geqq l_n^+$. Since there are only two more constraints than variables, we are able to explicitly find λ_n^+, and the structure of the extremum shows that $\lambda_n^+ - 2 < l_n^+$. Finally, we analyse the asymptotic behaviour of $\frac{\lambda_n^+}{n^{\frac{1}{2}}}$. Since obviously $l_n^-(\alpha) = l_n^+(1-\alpha)$, the results for l_n^- follow automatically.

2. One-sided diophantine approximation

If α is an irrational, we will denote its continued fraction expansion by $\alpha=[a_0; a_1, a_2, \dots]$, so that if $\alpha=a_0+\alpha_0$ with $a_0=[\alpha]$, $\alpha_0=\{\alpha\}$, then for $n=1,2,\dots$

$$a_n=\left[\frac{1}{\alpha_{n-1}}\right] \text{and } \alpha_n=\left\{\frac{1}{\alpha_{n-1}}\right\}.$$

If we write $\frac{p_n}{q_n}=[a_0; a_1, \dots, a_n]$ for the principal convergents, then [10], pp. 1—8,

$$(2) \qquad a_n+\alpha_n=\frac{1}{\alpha_{n-1}},$$

$$(3) \qquad q_n\alpha-p_n=(-1)^n(q_{n+1}+\alpha_{n+1}q_n)^{-1},\ q_{-1}=0,\ q_0=1,$$

$$(4) \qquad q_{n+1}=a_{n+1}q_n+q_{n-1}, \text{ for } n=0,1,\dots .$$

We shall write $\sigma_n=|q_n\alpha-p_n|$, so that $\sigma_n=||q_n\alpha||$ for $n>1$, where, as usual, $||x||=\min(\{x\}, 1-\{x\})$. Then (2), (3) and (4) imply

$$(5) \qquad \sigma_n=(q_{n+1}+\alpha_{n+1}q_n)^{-1},$$

$$(6) \qquad \sigma_{n+1}=\alpha_{n+1}\sigma_n,\quad n=0,1,\dots .$$

We define the intermediate convergents $\frac{p_{n,k}}{q_{n,k}}$ by

$$(7) \qquad p_{n,k}=kp_{n+1}+p_n,\quad 0<k<a_{n+2},$$

$$(8) \qquad q_{n,k}=kq_{n+1}+q_n,\quad 0<k<a_{n+2}.$$

Note that $q_{n,0}=q_n$ and $q_{n,a_{n+2}}=q_{n+2}$. Defining $\sigma_{n,k}=|q_{n,k}\alpha-p_{n,k}|$, (3) and (6) imply that

$$(9) \qquad \sigma_{n,k}=\sigma_n-k\sigma_{n+1}, \text{ for } 0\leqq k\leqq a_{n+2}.$$

The sequence q_n is characterized by the following well-known result:

Lemma 1. *For $n\geqq 2$, q_n is the smallest integer $q>q_{n-1}$ such that $||q\alpha||<||q_{n-1}\alpha||$.*

Proof. See [10], p. 10 or [1], p. 2.

We define $\{q_n^+\}$ to be the sequence of even-ordered denominators q_{2m} and $q_{2m,k}$ $(1\leqq k\leqq a_{2m+2}-1)$ arranged in increasing order $q_0<q_{0,1}<\dots<q_2<\cdots$, and $\{q_n^-\}$ to be the corresponding sequence of odd-ordered denominators. The following is an analogue of Lemma 1 which we have been unable to find explicitly stated in the standard literature:

Lemma 2. (i) *For $n\geqq 1$, q_n^+ is the smallest integer $q>q_{n-1}^+$ such that $\{q\alpha\}<\{q_{n-1}^+\alpha\}$.*

(ii) *For $n\geqq 1$, q_n^- is the smallest integer $q>q_{n-1}^-$ such that $\{q_{n-1}^-\alpha\}<\{q\alpha\}$.*

Proof. By (9), $\{q_n^+\alpha\}$ forms a decreasing sequence. Let q satisfy $q>q_{n-1}^+=q_{2m,k}$ and $\{q\alpha\}<\{q_{n-1}^+\alpha\}$. Write $q=q_{n-1}^++r$. Then, in order for $\{q_{n-1}^+\alpha+r\alpha\}<\{q_{n-1}^+\alpha\}$, we must have

$$(10) \qquad \{q\alpha\}=\{q_{n-1}^+\alpha\}+\{r\alpha\}-1=\{q_{n-1}^+\alpha\}-||r\alpha||.$$

Thus $||r\alpha||<\{q_{n-1}^+\alpha\}=\{q_{2m,k}\alpha\}\leqq\{q_{2m}\alpha\}=||q_{2m}\alpha||$. By Lemma 1, this means that $r\geqq q_{2m+1}$ so that $q\geqq q_{2m,k}+q_{2m+1}=q_{2m,k+1}=q_n^+$.

To prove (ii), apply (i) to $1-\alpha$.

Remark. The q_n^+ are the denominators in the semi-regular continued fraction defined by $\alpha = b_0 \frac{+1}{b_1} \frac{-1}{b_2} \frac{-1}{b_3} \cdots$, with $b_n \geqq 2$ for all $n \geqq 1$.

The easiest way to see this is to show that these denominators satisfy Lemma 2. It also follows from a result of Tietze [12], p. 163.

3. An integer programming problem

It is clear that $l_n^-(\alpha) = l_n^+(1-\alpha)$ so we will limit ourselves in this section to l_n^+.

Lemma 3. *Let α and n be fixed. Let q_m^+ be defined as in the previous section, and let M satisfy $q_M^+ \leqq n < q_{M+1}^+$. Then $l_n^+(\alpha)$ is the solution to the following:*

$$l_n^+ = \max \sum_{m=1}^{M} r_m, \tag{11}$$

where the r_i are integers for which

$$\sum_{m=1}^{M} r_m q_m^+ \leqq n, \tag{12a}$$

$$\sum_{m=1}^{M} r_m \{q_m^+ \alpha\} \leqq 1, \tag{12b}$$

$$r_1 \geqq 0, \ldots, r_M \geqq 0. \tag{12c}$$

Proof. Suppose $0 < \{n_1\alpha\} < \cdots < \{n_l\alpha\} < 1$ is an increasing sequence of length l with $1 \leqq n_1 < \cdots < n_l \leqq n$. If we write $d_1 = n_1$, $d_k = n_k - n_{k-1}$ for $k = 2, \ldots, l$, then we obtain $\{d_k\alpha\} = \{n_k\alpha\} - \{n_{k-1}\alpha\}$, and we have

$$\sum_{k=1}^{l} \{d_k\alpha\} \leqq 1 \quad \text{and} \quad \sum_{k=1}^{l} d_k \leqq n. \tag{13}$$

By definition, l_n^+ is the maximum value of l under these constraints.

We claim that, in (13), it is no restriction to choose d_k from among the q_m^+. For, if $q_m^+ < d_k < q_{m+1}^+$, then $\{d_k\alpha\} > \{q_m^+\alpha\}$, by Lemma 2, so if d_k is replaced by q_m^+, both inequalities in (13) continue to hold. But then, collecting terms in q_m^+ and $\{q_m^+\alpha\}$, (13) reduces to (12).

4. A linear programming problem

Let $\lambda_n^+(\alpha)$ denote the solution to the problem defined by (11) and (12), where r_m and n are no longer required to be integers. Define $\lambda_n^-(\alpha) = \lambda_n^+(1-\alpha)$. Then obviously $\lambda_n^+ \geqq l_n^+$ and $\lambda_n^- \geqq l_n^-$. Now that real coefficients are to be allowed, we find that the intermediate denominators $q_{2m,k}$ which appear in q_m^+ are no longer needed:

Lemma 4. *Let K satisfy $q_{2K} \leqq n < q_{2K+2}$. Then*

$$\lambda_n^+ = \max \sum_{k=1}^{K} x_k,$$

where $(x_1, \ldots, x_K)$ *satisfies*

$$\sum_{k=1}^{K} x_k q_{2k} \leqq n, \tag{14a}$$

$$\sum_{k=1}^{K} x_k \sigma_{2k} \leqq 1, \tag{14b}$$

$$x_1 \geqq 0, \ldots, x_K \geqq 0. \tag{14c}$$

Proof. By (8) and (9), $q_{2m,k} = (1-c)q_{2m} + cq_{2m+2}$ and $\sigma_{2m,k} = (1-c)\sigma_{2m} + c\sigma_{2m+2}$, where $c = \dfrac{k}{a_{2m+2}}$. Thus, in (12a), we may replace any q_i^+ of the form $q_{2m,k}$ by a convex combination of q_{2m} and q_{2m+2}, and in (12b), replace $\{q_{2m,k}\alpha\} = \sigma_{2m,k}$ by the same combination of σ_{2m} and σ_{2m+2}, without affecting the inequalities.

Theorem 1. *Define* $\beta_{-1} = (1 - \{\alpha\})^{-1}$ *and* $\beta_m = q_m \sigma_m^{-1}$, $m = 0, 1, \ldots$. *Then, for any given* α, λ_n^+ *and* λ_n^- *are the following piecewise linear functions of* n:

$$\lambda_n^+ = \begin{cases} q_{2k+1} + n\sigma_{2k+1}, & \textit{if } \beta_{2k} \leqq n < \beta_{2k+2},\ k = 0, 1, \ldots, \\ n, & \textit{if } 0 \leqq n < \beta_0, \end{cases} \tag{15}$$

$$\lambda_n^- = \begin{cases} q_{2k} + n\sigma_{2k}, & \textit{if } \beta_{2k-1} \leqq n < \beta_{2k+1},\ k = 0, 1, \ldots, \\ n, & \textit{if } 0 \leqq n < \beta_{-1}. \end{cases} \tag{16}$$

Proof. We begin with (15). The constraint region (14) has $\binom{K+2}{K}$ possible vertices. Apart from $(0, \ldots, 0)$, either $K-1$ or $K-2$ of the x_k must be 0. If all x_k but x_i are 0, which we call a vertex of type I, then (14a), (14b) give

$$x_i = \min(\sigma_{2i}^{-1}, nq_{2i}^{-1}). \tag{17}$$

If all x_k but x_i and x_j are zero, called a vertex of type II, then x_i, x_j must solve the equations

$$x_i \sigma_{2i} + x_j \sigma_{2j} = 1, \tag{18a}$$

$$x_i q_{2i} + x_j q_{2j} = n \tag{18b}$$

so that

$$x_i = \sigma_{2i}^{-1} \frac{\beta_{2j} - n}{\beta_{2j} - \beta_{2i}}, \tag{19a}$$

$$x_j = \sigma_{2j}^{-1} \frac{n - \beta_{2i}}{\beta_{2j} - \beta_{2i}}. \tag{19b}$$

Assuming $i < j$, then $\sigma_{2i}^{-1} < \sigma_{2j}^{-1}$ and $\beta_{2i} < \beta_{2j}$, so the condition for $x_i \geqq 0$, $x_j \geqq 0$ is seen to be $\beta_{2i} \leqq n \leqq \beta_{2j}$. Among the vertices of types I and II, we seek to maximize $\lambda = x_1 + \cdots + x_K$. Let us denote the right member of (17) by $f_i(n)$ and by $g_{i,j}(n)$ the value of $\lambda = x_i + x_j$ given by (19). Then

$$\lambda_n^+ = \max(\max_i f_i(n), \max_{i,j} g_{i,j}(n)), \tag{20}$$

where the maximum over i, j is restricted to those which satisfy $\beta_{2i} \leqq n \leqq \beta_{2j}$.

Define v_i to be the point $(\beta_{2i}, \sigma_{2i}^{-1})$, $i=0,1,\dots$ and $v_{-1}=(0,0)$. Then the graph of $f_i(n)$ is the line segment $v_{-1}v_i$ followed by the horizontal line from v_i to ∞. The graph of $g_{i,j}(n)$ is the line segment from v_i to v_j. Let us denote the right member of (15) by $f(n)$. Then, in fact, the graph of $f(n)$ is the polygonal line $v_{-1}v_0v_1\dots$. To see this, observe that

$f(\beta_{2k})=q_{2k+1}+\beta_{2k}\sigma_{2k+1}=q_{2k+1}+q_{2k}\sigma_{2k}^{-1}\sigma_{2k+1}=q_{2k+1}+q_{2k}\alpha_{2k+1}=\sigma_{2k}^{-1}$, by (5) and (6). Also $f(\beta_{2k+2}-)=q_{2k+1}+\beta_{2k+2}\sigma_{2k+1}=\sigma_{2k+2}^{-1}$, by a similar calculation. The function f is thus continuous, increasing and concave, since the slope σ_{2k+1} decreases with k. Thus, the graphs of $f_i(n)$ and $g_{i,j}(n)$ lie strictly below the graph of $f(n)$, except for $f_0(n)$ and $g_{i,i+1}(n)$, $i=0,1,\dots$ which coincide with $f(n)$ on the intervals $[0,\beta_0]$, $[\beta_{2i},\beta_{2i+2}]$ respectively. This shows that $\lambda_n^+=f(n)$.

To prove (16), one uses $\lambda_n^-(\alpha)=\lambda_n^+(1-\alpha)$. The two cases $\alpha>\frac{1}{2}$ and $\alpha<\frac{1}{2}$ need to be distinguished. If $\alpha>\frac{1}{2}$, note that $\beta_{-1}=\beta_1$.

Corollary 1. *For all* α, $\lambda_n^+-2<l_n^+\leqq\lambda_n^+$ *and* $\lambda_n^- -2<l_n^-\leqq\lambda_n^-$.

Proof. Clearly $\lambda_n^+\geqq l_n^+$. On the other hand, by the proof of Theorem 1, if $(x_1,\dots,x_K)$ satisfies (14) and has $\lambda_n^+=x_1+\dots+x_K$, then $(x_1,\dots,x_K)$ need only have two non-zero components. Thus

$$l_n^+\geqq\sum_{k=1}^{K}[x_k]>\left(\sum_{k=1}^{K}x_k\right)-2=\lambda_n^+-2.$$

The proof for λ_n^- follows by the standard symmetry.

5. The asymptotic behaviour of l_n^+ and l_n^-

Theorem 2. *The sequence* $\dfrac{\lambda_n^+}{n^{\frac{1}{2}}}$ *oscillates between local maxima of size* $\dfrac{1}{(q_{2k}\sigma_{2k})^{\frac{1}{2}}}$ *attained at* $n=\beta_{2k}$ *and local minima of size* $2(q_{2k+1}\sigma_{2k+1})^{\frac{1}{2}}$ *attained at* $n=\beta_{2k+1}$. *In a self explanatory notation,*

(21 a) $$\text{loc max}\,\frac{\lambda_n^+}{n^{\frac{1}{2}}}=\frac{1}{(q_{2k}\sigma_{2k})^{\frac{1}{2}}}$$

(21 b) $$\text{loc min}\,\frac{\lambda_n^+}{n^{\frac{1}{2}}}=2(q_{2k+1}\sigma_{2k+1})^{\frac{1}{2}}.$$

In the same notation,

(22 a) $$\text{loc max}\,\frac{\lambda_n^-}{n^{\frac{1}{2}}}=\frac{1}{(q_{2k+1}\sigma_{2k+1})^{\frac{1}{2}}}$$

(22 b) $$\text{loc min}\,\frac{\lambda_n^-}{n^{\frac{1}{2}}}=2(q_{2k}\sigma_{2k})^{\frac{1}{2}}.$$

Proof. By (15), if $\beta_{2k} \leqq n \leqq \beta_{2k+2}$ we have

$$\frac{\lambda_n^+}{n^{\frac{1}{2}}} = q_{2k+1} n^{-\frac{1}{2}} + \sigma_{2k+1} n^{\frac{1}{2}}$$

which, as a function of n, decreases to a minimum of $2(q_{2k+1}\sigma_{2k+1})^{\frac{1}{2}}$ at $n = \beta_{2k+1}$, and attains a local maximum at $n = \beta_{2k}$ equal to

$$q_{2k+1} q_{2k}^{-\frac{1}{2}} \sigma_{2k}^{\frac{1}{2}} + \sigma_{2k+1} q_{2k}^{\frac{1}{2}} \sigma_{2k}^{-\frac{1}{2}} = \sigma_{2k}^{\frac{1}{2}} q_{2k}^{-\frac{1}{2}} (q_{2k+1} + \alpha_{2k+1} q_{2k}) = \frac{1}{(q_{2k}\sigma_{2k})^{\frac{1}{2}}},$$

using (5) and (6). This proves (21), and (22) follows as usual.

Corollary 2. *Let* $A = \liminf q_{2k}\sigma_{2k}$ *and* $B = \liminf q_{2k+1}\sigma_{2k+1}$. Then

$$\limsup \frac{l_n^+}{n^{\frac{1}{2}}} = A^{-\frac{1}{2}}, \qquad \liminf \frac{l_n^+}{n^{\frac{1}{2}}} = 2B^{\frac{1}{2}},$$

$$\limsup \frac{l_n^-}{n^{\frac{1}{2}}} = B^{-\frac{1}{2}}, \qquad \liminf \frac{l_n^-}{n^{\frac{1}{2}}} = 2A^{\frac{1}{2}}.$$

Proof. A direct consequence of (21) and (22).

Remarks. The quantity $q_k\sigma_k = q_k\|q_k\alpha\|$ appears naturally in the theory of rational approximation, as does the quantity $\nu(\alpha) = \liminf q_k\|q_k\alpha\|$ [1], p. 11. The set of values $\nu(\alpha)$ is called the Lagrange spectrum. By a theorem of Hurwitz, the largest value of $\nu(\alpha)$ is $5^{-\frac{1}{2}}$. The proof of Corollary 4, below, is modelled on Davenport's proof of the theorem of Hurwitz, as given in [1], p. 11.

Two real numbers α, α' are said to be equivalent if $\alpha = \frac{a\alpha' + b}{c\alpha' + d}$, where a, b, c, d are integers with $ad - bc = \pm 1$. A necessary and sufficient condition for α and α' to be equivalent is that their continued fraction expansions can be shifted so as to coincide beyond some point [1], p. 9.

Using (5), we have the following convenient representation:

$$(23) \qquad (q_k\sigma_k)^{-1} = a_{k+1} + \alpha_{k+1} + \frac{q_{k-1}}{q_k} = [a_{k+1}; a_{k+2}, \ldots] + [0; a_k, a_{k-1}, \ldots, a_1].$$

Thus, for example, if $\alpha = \tau = \frac{5^{\frac{1}{2}} + 1}{2} = [1; 1, 1, \ldots]$, then

$$(q_k\sigma_k)^{-1} = [1; 1, 1, \ldots] + [0; 1, \ldots, 1] \to \frac{5^{\frac{1}{2}} + 1}{2} + \frac{5^{\frac{1}{2}} - 1}{2} = 5^{\frac{1}{2}}.$$

Thus $\limsup \frac{l_n^+}{n^{\frac{1}{2}}} = \limsup \frac{l_n^-}{n^{\frac{1}{2}}} = 5^{\frac{1}{4}} = 1.495349\ldots$, while $\liminf \frac{l_n^+}{n^{\frac{1}{2}}} = \liminf \frac{l_n^-}{n^{\frac{1}{2}}} = 1.337481\ldots$.

This is an extreme example as the following shows:

Corollary 3. *Let* $a = \limsup \frac{l_n^+}{n^{\frac{1}{2}}}$ *and* $b = \liminf \frac{l_n^+}{n^{\frac{1}{2}}}$. *Then* $0 < b$ *and* $a < \infty$ *if and only if* $\{a_n\}$ *is a bounded sequence. There is no* α *for which* $a = b$, *and in fact the minimum values of* $\frac{a}{b}$ *and* $a - b$ *are* $\frac{5^{\frac{1}{2}}}{2}$ *and* $5^{\frac{1}{4}} - \frac{2}{5^{\frac{1}{4}}}$ *attained only for* $\alpha = \frac{5^{\frac{1}{2}}+1}{2}$ *and numbers equivalent to it.*

Proof. The first statement is obvious since, by (23)

$$a_{k+1} < (q_k \sigma_k)^{-1} < a_{k+1} + 2. \tag{24}$$

To prove the second statement, let $A_k = q_k \sigma_k$. Then the following inequality may be derived from (2), (4) and (5), [1], p. 12:

$$a_{k+1}^2 A_k^2 + 2 A_k (A_{k-1} + A_{k+1}) < 1. \tag{25}$$

Suppose that an infinite set of a_m satisfy $a_m \geqq 2$. To be specific, consider the case where $a_{2k} \geqq 2$ for an infinite set of k. Then (25) implies, for these k, that

$$4 A_{2k-1}^2 + 2 A_{2k-1}(A_{2k-2} + A_{2k}) < 1,$$

and

$$A_{2k}^2 + 2 A_{2k}(A_{2k+1} + A_{2k-1}) < 1.$$

Thus

$$4 B^2 + 4 AB \leqq 1, \tag{26}$$

$$A^2 + 4 AB \leqq 1. \tag{27}$$

We may rewrite (26) and (27) as

$$\frac{B}{A} \leqq (4AB)^{-1} - 1, \tag{28}$$

$$\frac{A}{4B} \leqq (4AB)^{-1} - 1. \tag{29}$$

Taking the geometric mean of (28) and (29), we find that $(4AB)^{-1} - 1 \geqq \frac{1}{2}$, so $(4AB)^{-1} \geqq \frac{3}{2}$ which implies that $\frac{a}{b} = (4AB)^{-\frac{1}{2}} \geqq \left(\frac{3}{2}\right)^{\frac{1}{2}} > \frac{5^{\frac{1}{2}}}{2}$. A similar proof applies if an infinite number of a_{2k+1} are $\geqq 2$. Thus, only if all but a finite number of a_m are equal to 1 we will have $\frac{a}{b} = \frac{5^{\frac{1}{2}}}{2}$, and this is in fact attained by such α as we saw above.

Furthermore $a \geqq 1$ for all α. So, if α is not equivalent to τ then

$$a-b=a\left(1-\left(\frac{2}{3}\right)^{\frac{1}{2}}\right)+\left(\left(\frac{2}{3}\right)^{\frac{1}{2}}a-b\right)\geqq a\left(1-\left(\frac{2}{3}\right)^{\frac{1}{2}}\right)\geqq 1-\left(\frac{2}{3}\right)^{\frac{1}{2}}=.1835\ldots,$$

while for numbers equivalent to τ, we have

$$a-b=5^{\frac{1}{4}}-\frac{2}{5^{\frac{1}{4}}}=.1578\ldots.$$

Example. We see from (21), (22) and (24) that loc max $\frac{\lambda_n^+}{n^{\frac{1}{2}}}\cong a_{2k+1}^{\frac{1}{2}}$ and loc max $\frac{\lambda_n^-}{n^{\frac{1}{2}}}\cong a_{2k}^{\frac{1}{2}}$ so the upper extremes of $\frac{l_n^+}{n^{\frac{1}{2}}}$ are governed by the partial quotients of odd order, while those of $\frac{l_n^-}{n^{\frac{1}{2}}}$ are governed by the partial quotients of even order. The following is thus an instructive example:

$$\alpha=\tan\left(\frac{1}{2}\right)=[0;1,1,4,1,8,1,12,1,16,\ldots],$$

with $a_{2k}=1$ and $a_{2k+1}=4k$ if $k\geqq 1$. This is obtained from Lambert's expansion of $\tan\left(\frac{1}{2}\right)$ in a semi-regular continued fraction [12], p. 353 by the use of Lagrange's transformation [12], p. 159. Using (23), we have $A=0$, $B=1$ so that

$$\limsup\frac{l_n^+}{n^{\frac{1}{2}}}=\infty,\quad \liminf\frac{l_n^+}{n^{\frac{1}{2}}}=2,$$

$$\limsup\frac{l_n^-}{n^{\frac{1}{2}}}=1,\quad \liminf\frac{l_n^-}{n^{\frac{1}{2}}}=0.$$

In fact, one can be more precise about $\frac{l_n^+}{n^{\frac{1}{2}}}$. By making the obvious estimates in

$$q_{2k+1}=4k\,q_{2k}+q_{2k-1},\quad q_{2k}=q_{2k-1}+q_{2k-2},$$

one has $4^k k!<q_{2k+1}<8^k(k+1)!$, so $q_{2k+2}\cong q_{2k+1}\cong(ck)^k$ (compare the discussion for e given in [10], p. 78). If $\beta_{2k}\leqq n<\beta_{2k+2}$, then

$$q_{2k}(q_{2k+1}+\alpha_{2k+1}q_{2k})\leqq n<q_{2k+2}(q_{2k+3}+\alpha_{2k+1}q_{2k+2}),$$

so $n\cong(ck)^k$ and hence $k\cong\frac{\log n}{\log\log n}$. Hence

$$\text{loc max}\,\frac{l_n^+}{n^{\frac{1}{2}}}\cong(4k)^{\frac{1}{2}}\cong\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{2}}.$$

The metric theory of continued fractions gives us similarly precise information for almost all α. The following result is sufficient for our purposes:

Lemma 5 (E. Borel and F. Bernstein [9], p. 63). *Let $\phi(k)$ be an increasing function of k. If $\sum_{k=1}^{\infty} \phi(k)^{-1}=\infty$ then for almost all α, $a_k \geqq \phi(k)$ for infinitely many k. On the other hand, if $\sum_{k=1}^{\infty} \phi(k)^{-1}<\infty$, then for almost all α, $a_k \leqq \phi(k)$ except for a finite set of k.*

Corollary 4. *Let $l_n^{\pm}$ denote either l_n^+ or l_n^-. For almost all α, and any $\varepsilon>0$, there are constants $C>0$ and $D<\infty$ so that for all n,*

$$Cn^{\frac{1}{2}}(\log n)^{-\frac{1}{2}-\varepsilon} \leqq l_n^{\pm} \leqq Dn^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\varepsilon}. \tag{30}$$

On the other hand

$$l_n^{\pm} \leqq Cn^{\frac{1}{2}}(\log n)^{-\frac{1}{2}+\varepsilon} \quad \text{and} \quad l_n^{\pm} \geqq Dn^{\frac{1}{2}}(\log n)^{\frac{1}{2}-\varepsilon}$$

hold for infinitely many n.

Proof. For almost all α, $q_k \cong c^k$ for a universal constant c [9], p. 66. Thus $\beta_{2k} \leqq n$ implies that $n \geqq c_1^k$ so $k \leqq c_2 \log n$. By Lemma 5, for almost all α, $a_k \leqq c_3 k^{1+2\varepsilon}$ and thus, by (21 a) and (24),

$$l_n^+ \leqq \lambda_n^+ \leqq n^{\frac{1}{2}}(q_{2k}\sigma_{2k})^{-\frac{1}{2}} \leqq n^{\frac{1}{2}}(a_{2k+1}+2)^{\frac{1}{2}} \leqq Dn^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\varepsilon}.$$

The remainder of the proof is similar.

Remark. Of course $(\log n)^{\frac{1}{2}+\varepsilon}$ can be replaced by $\psi(\log n)$ for any increasing ψ for which $\sum \psi(k)^{-2}<\infty$.

Corollary 5. *For all α, $n \leqq l_n^+ l_n^- \leqq 2n$, and* $\limsup \frac{l_n^+ l_n^-}{n}=2$. *If α has unbounded partial quotients, then* $\liminf \frac{l_n^+ l_n^-}{n}=1$. *For all α,*

$$\liminf \frac{l_n^+ l_n^-}{n} \leqq 1+\frac{2}{5^{\frac{1}{2}}}=1.894427\ldots, \tag{31}$$

which is attained only for $\alpha=\frac{5^{\frac{1}{2}}+1}{2}$ and equivalent numbers.

Proof. The fact that $l_n^+ l_n^- \geqq n$ is a result of Erdös and Szekeres [4]. By Theorem 1, if $\beta_m \leqq n \leqq \beta_{m+1}$, then

$$\lambda_n^+ \lambda_n^- = g(n) = (q_m + n\sigma_m)(q_{m+1}+n\sigma_{m+1}).$$

By (5) and (6), $g(\beta_m)=2\beta_m$ and $g(\beta_{m+1})=2\beta_{m+1}$. Thus $g(n)-2n$ is quadratic in n and vanishes at $n=\beta_m, \beta_{m+1}$ and hence satisfies $g(n)<2n$ for $\beta_m<n<\beta_{m+1}$. Hence

$$l_n^+ l_n^- \leqq \lambda_n^+ \lambda_n^- \leqq 2n.$$

Since $\lambda_n^+ \lambda_n^- = 2n$ for $n=\beta_m$, it follows that $\limsup \frac{l_n^+ l_n^-}{n} = \limsup \frac{\lambda_n^+ \lambda_n^-}{n} = 2$.

On the other hand $\frac{g(n)}{n}$ has a minimum at $n=(\beta_m\beta_{m+1})^{\frac{1}{2}}$. Using $q_{m+1}\sigma_m+q_m\sigma_{m+1}=1$, we have

$$\text{loc min}\ \frac{\lambda_n^+\lambda_n^-}{n}=1+2(A_mA_{m+1})^{\frac{1}{2}},$$

where $A_m=q_m\sigma_m$. Thus

$$\liminf \frac{l_n^+ l_n^-}{n}=1+\liminf 2(A_mA_{m+1})^{\frac{1}{2}}, \tag{32}$$

which is 1 if $\{a_m\}$ is unbounded. Let $c_m=A_mA_{m+1}$. Then (25) becomes

$$a_{m+1}^2A_m^2<1-2c_{m-1}-2c_m. \tag{33}$$

Thus

$$a_{m+1}^2a_{m+2}^2c_m^2<(1-2c_{m-1}-2c_m)(1-2c_m-2c_{m+1}). \tag{34}$$

Let $c=\liminf c_m$, which is clearly at most $\frac{1}{4}$ by (33). If $a_m\geqq 2$ for infinitely many m, then (34) shows that $4c^2\leqq(1-4c)^2$, so $c\leqq\frac{1}{6}$. Thus $\liminf\frac{l_n^+l_n^-}{n}\leqq 1+\frac{2}{6^{\frac{1}{2}}}<1+\frac{2}{5^{\frac{1}{2}}}$, from (32). It is easy to check that equality holds in (31) for any α equivalent to τ.

Added in proof. With respect to Lemma 2 we remark that J. L. Nicolas states that result without proof in: Répartition Modulo 1, Lecture notes in Mathematics **475,** New York 1975, p. 115.

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