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WORST-CASE GROWTH RATES OF SOME CLASSICAL PROBLEMS OF COMBINATORIAL OPTIMIZATION*

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Abstract. A method is presented for determining the asymptotic worst-case behavior of quantities like the length of the minimal spanning tree or the length of an optimal traveling salesman tour of n points in the unit d-cube. In each of these classical problems, the worst-case lengths are proved to have the exact asymptotic growth rate of $\beta_n^{(d-1)/d}$ as $n \to \infty$, where β is a positive constant depending on the problem and the dimension. These results complement known results on the growth rates for the analogous quantities under probabilistic assumptions on the points, but the results given here are free of any probabilistic hypotheses.

Key words. asymptotics, traveling salesman problem, minimal spanning tree, Beardwood-Halton-Hammersley theorem

AMS(MOS) subject classifications. 05C35, 90B10, 90C10, 52A40

1. Introduction. The purpose of this paper is to illustrate a general method for determining the asymptotic behavior of some classical quantities of operations research and combinatorial optimization. For specificity, we focus on the traveling salesman problem and on the minimal spanning tree of n points in the unit d-cube, but the general applicability of our method to a number of other problems will be made evident.

To set our problem precisely, we first note that a Euclidean minimal spanning tree or a traveling salesman tour can be represented by a graph $G = (V_n, E)$, where V_n denotes a set of n points in $[0, 1]^d$, where $d \geqq 2$, and E denotes a subset of the edges of the complete graph on the points of V_n. The length of an edge $e = \{x_i, x_j\}$ is taken to be the usual Euclidean distance, and we write $|e| = |x_i - x_j|$ for that length. For a collection E of edges we will often use $L(E)$ to denote the sum of the lengths of the edges in E, i.e., we define $L(E) = \sum_{e \in E} |e|$. Still, when V is a finite set there will be no ambiguity in using $|V|$ to denote the cardinality of V.

The objects of principal interest here are the sequences $\rho_{\text{MST}}(n)$ and $\rho_{\text{TSP}}(n)$, defined by

$$\rho_{\text{MST}}(n) = \max_{\substack{V_n \subset [0,1]^d \\ |V_n| = n}} \left\{ \min_T \sum_{e \in T} |e| \colon T \text{ is a spanning tree of } V_n \right\}$$

and

$$\rho_{\text{TSP}}(n) = \max_{\substack{V_n \subset [0,1]^d \\ |V_n| = n}} \left\{ \min_T \sum_{e \in T} |e| \colon T \text{ is a tour of } V_n \right\}.$$

In other words, $\rho_{\text{MST}}(n)$ is equal to the largest possible length of any minimal spanning tree formed from n points in $[0, 1]^d$. Similarly, $\rho_{\text{TSP}}(n)$ is the largest possible length of any optimal traveling salesman tour through n points in $[0, 1]^d$. The use of max instead of sup in the definitions of $\rho_{\text{MST}}(n)$ and $\rho_{\text{TSP}}(n)$ is justified by the fact that the expressions in braces can be viewed as continuous functions on the compact set obtained by forming the product of n copies of $[0, 1]^d$, i.e., $\prod_{1 \leqq i \leqq n} [0, 1]^d$.

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One should note that the functions ρ_{MST} and ρ_{TSP} depend on the dimension d. This fact also applies to all of the other functions and constants that are used here. Since $d \geqq 2$ is fixed, we will suppress the dependence of ρ_{MST}, ρ_{TSP}, and other functions on d, but the reader should be mindful of this dependence, especially in the main result.

THEOREM. *There are constants β_{MST} and β_{TSP} depending on the dimension $d \geqq 2$ such that*

$$\rho_{MST}(n) \sim \beta_{MST} n^{(d-1)/d} \tag{1.1}$$

and

$$\rho_{TSP}(n) \sim \beta_{TSP} n^{(d-1)/d} \tag{1.2}$$

as $n \to \infty$, with $\beta_{TSP} \geqq \beta_{MST} \geqq 1$.

This result provides the determination of the exact asymptotic order of the functions ρ_{MST} and ρ_{TSP} in any dimension $d \geqq 2$. Considerable earlier effort focused on bounds for $\rho_{MST}(n)$ and $\rho_{TSP}(n)$, but none of the inequalities provided by that work is tight enough to determine that $\rho_{MST}(n)$ or $\rho_{TSP}(n)$ are actually asymptotic to a constant times $n^{(d-1)/d}$. Some earlier results of particular interest are the bound of Verblunsky (1951), which says that in $d=2$ one has $\rho_{TSP}(n) \leqq (2.8n)^{1/2} + 3.15$, and the bounds of Fejes-Tóth (1940), which say that $\rho_{TSP}(n)$ and $\rho_{MST}(n)$ are both at least as large as $(1-\varepsilon)(4/3)^{1/4} n^{1/2}$ for all $n \geqq N(\varepsilon)$. Few (1955) improved the upper bound of Verblunsky (1951) to $\rho_{TSP}(n) \leqq (2n)^{1/2} + 1.75$ in $d=2$ and obtained $\rho_{TSP}(n) \leqq d\{2(d-1)\}^{(1-d)/2d} n^{(d-1)/d} + 0(n^{1-2/d})$ for general $d \geqq 2$.

Recent results have improved these bounds. Few's bound on $\rho_{TSP}(n)$ in dimension two is sharpened in Supowit, Reingold, and Plaisted (1983), to show that $\rho_{TSP}(n) \geqq (4/3)^{1/4} n^{1/2}$, for all $n \geqq 1$. Moran (1984) used inequalities on sphere packing to obtain essential improvements on the upper bounds of Few for large values of d. Goldstein and Reingold (1988) carefully analyze Few's heuristic algorithm to improve the upper bounds in dimensions $3 \leqq d \leqq 7$. They also improve lower bounds, using the exact densities of sphere packings for $2 \leqq d \leqq 8$. Goldstein (personal communication) has further improved the upper bounds in dimensions three and four.

The $(2n)^{1/2}$ barrier on $\rho_{TSP}(n)$ in dimension two is broken by bounds of Karloff (1987) that show $\rho_{TSP}(n) < 0.984(2n)^{1/2} + 11$. Also, for low dimensions $d \geqq 3$, Goddyn (1988) improves all known upper bounds on $\rho_{TSP}(n)$ by considering an infinite number of translations of quantizers other than cubical cylinders.

Some other early work focused on the probabilistic circumstances under which one can provide bounds for the lengths of the minimal spanning tree or optimal traveling salesman tour. For example, Ghosh (1949) sharpened earlier results of Mahalanobis (1940) and Jessen (1942) to establish that the expected length of an optimal traveling salesman tour of n points chosen at random from the unit square was at most $1.27n^{1/2} + 0(1)$. The bound of Marks (1948) complements the upper bound of Ghosh (1949) by providing a lower bound of $(n^{1/2} - 1/n^{1/2})/2$ on the expected length of an optimal traveling salesman tour in $d=2$.

The culminating result on the length of an optimal traveling salesman tour under probabilistic assumptions was provided by Beardwood, Halton, and Hammersley (1959). That work showed that if T_n denotes the length of an optimal traveling salesman tour of X_i, where $1 \leqq i \leqq n$ and the X_i are bounded independent identically distributed random vectors in $\mathbb{R}^d$, then with probability one we have the asymptotic relation

$$T_n \sim c_d n^{(d-1)/d} \int_{\mathbb{R}^d} f(x)^{(d-1)/d}\, dx. \tag{1.3}$$

Here, f denotes the density of the absolutely continuous part of the distribution of the X_i, and c_d is a constant depending only on the dimension.

In addition to providing improved upper and lower bounds on the constant c_d, Beardwood, Halton, and Hammersley (1959) also indicated that a result analogous to (1.3) holds for the minimal spanning tree. A review of the probability theory which has grown out of the Beardwood, Halton, and Hammersley theorem is given in Steele (1987), and a review oriented toward algorithmic applications is given in Karp and Steele (1985).

The focus of the present work is on the growth rates of the *worst-case* lengths of the traveling salesman tour and minimal spanning tree. There are no probabilistic assumptions used here, and it is perhaps remarkable that one obtains asymptotics that are so close in form to the probabilistic results. Another intriguing aspect of these limit theorems is that the same method applies both to a computationally difficult problem (the TSP) and to one which is computationally easy (the MST).

The proof of the main theorem is given in three sections. The first of these sections provides a general lemma that isolates inequalities that are sufficient to determine the asymptotic behavior of ρ_{MST} and ρ_{TSP}. The following section focuses on minimal spanning trees, and, in particular, it provides an approximate recursion relation for ρ_{MST}. The construction used to study ρ_{TSP} in § 4 is much like that used for ρ_{MST}; so the analysis required for the optimal traveling salesman tour is quite brief.

The final section points out some limitations of this method and comments on some open problems.

2. Asymptotics from an approximate recursion. One principle underlying our asymptotic analysis is that both $\rho_{\mathrm{MST}}(n)$ and $\rho_{\mathrm{TSP}}(n)$ satisfy inequalities which bound their rates of growth and express an approximate recursiveness. The following lemma shows that a slow incremental rate of growth (as expressed by (2.1(i))) and an approximate recursiveness (as expressed by (2.1(ii))) are together sufficient to determine the exact asymptotic behavior of a sequence. Even though the lemma appears technical, we will later see that the two required conditions are quite natural to the objects under study.

LEMMA 2.1. *If* $\rho(1)=0$ *and there is a constant* $c_1 \geqq 0$ *such that for all* $m \geqq 1$ *and* $k \geqq 1$

$$\text{(i)}\quad \rho(n+1) \leqq \rho(n) + c_1 n^{-1/d}$$

(2.1) *and*

$$\text{(ii)}\quad m^{d-1}\rho(k) - m^{d-1}k^{(d-1)/d}r(k) \leqq \rho(m^d k),$$

where $r(k)\to 0$ *as* $k\to\infty$, *then as* $n\to\infty$

$$\rho(n) \sim \beta n^{(d-1)/d}$$

for a constant β.

Proof. From the hypothesis (2.1(i)) and the fact that $\rho(1)=0$ we first note that for $1 \leqq i < j < \infty$ we have

$$\begin{aligned}\rho(j)-\rho(i) &= \sum_{k=i}^{j-1}\{\rho(k+1)-\rho(k)\}\\ &\leqq c_1\int_{i-1}^{j-1} x^{-1/d}\,dx \leqq 5c_1\left(j^{(d-1)/d} - i^{(d-1)/d}\right).\end{aligned} \tag{2.2}$$

Letting $i=1$ and $j=n$ in (2.2) shows that $\rho(n) \leqq 5c_1 n^{(d-1)/d}$, so if we define $\psi(k) = \rho(k)/k^{(d-1)/d}$, then we see that $\psi(k) < 5c_1$ for all k. We can then introduce a candidate for our limit by

$$\gamma = \limsup_{k\to\infty} \psi(k) < \infty. \tag{2.3}$$

Inequality (2.1(ii)) tells us that for all k and m,

$$\psi(k) - r(k) \leqq \psi(m^d k), \tag{2.4}$$

so, given any fixed $\varepsilon > 0$, we can choose a k_ε such that $\gamma - \varepsilon \leqq \psi(k_\varepsilon)$ and $r(k_\varepsilon) \leqq \varepsilon$, thus obtaining from (2.4) that

$$\gamma - 2\varepsilon \leqq \psi(m^d k_\varepsilon) \tag{2.5}$$

for all $m \geqq 1$.

Next define $j_m = m^d k_\varepsilon$ and consider n such that $j_m \leqq n \leqq j_{m+1}$. To bound the absolute difference $|\psi(n) - \psi(j_m)|$, we use (2.2):

$$\begin{aligned} \sup_{j_m \leqq n \leqq j_{m+1}} |\rho(n) - \rho(j_m)| &\leqq 5c_1(j_{m+1}^{(d-1)/d} - j_m^{(d-1)/d}) \\ &= 5c_1 k_\varepsilon^{(d-1)/d} m^{d-1}[(1+1/m)^{d-1} - 1], \end{aligned} \tag{2.6}$$

or, in terms of ψ, the binomial expansion gives

$$\sup_{j_m \leqq n \leqq j_{m+1}} |\psi(n) - \psi(j_m)| \leqq 5c_1\{(1+1/m)^{d-1} - 1\} < 5c_1 m^{-1} 2^{d-1}. \tag{2.7}$$

From (2.7) and (2.5) we find for $j_m \leqq n \leqq j_{m+1}$ that

$$\gamma - 2\varepsilon - 5c_1 m^{-1} 2^{d-1} \leqq \psi(n),$$

and, hence, $\gamma - 2\varepsilon \leqq \liminf_{n\to\infty} \psi(n)$. By the arbitrariness of $\varepsilon > 0$, we have proved

$$\limsup_{n\to\infty} \psi(n) \leqq \liminf_{n\to\infty} \psi(n)$$

and the lemma is complete.

3. Minimal spanning trees. We will now show that ρ_{MST} satisfies the hypotheses of the preceding lemma. The key issue is the derivation of an inequality like (2.1(ii)). This will be done by a recursive construction of a point set for which a minimal spanning tree has near maximal length.

We first divide the d-cube $Q = [0, 1]^d$ into m^d *cells* Q_i, where $1 \leqq i \leqq m^d$ and each cell has side length $1/m$. The boundaries ∂Q_i of the cells Q_i create a natural *grating* in the unit d-cube which we denote by H, i.e., we set $H = \bigcup_{i=1}^{m^d} \partial Q_i$. For $0 < \alpha < 1/m$, let H^α denote the set of points of $[0, 1]^d$ which are within $\alpha/2$ of H, thus H^α is the grating H fattened to a width of α. Similarly, we define *subcells* Q_i^α of Q_i by $Q_i^\alpha = Q_i - H^\alpha$.

Inside each of the Q_i^α we now place a set S_i of k points for which the length of the minimal spanning tree is $(m^{-1} - \alpha)\rho_{\mathrm{MST}}(k)$, i.e., inside each subcell we place a copy of a set of k points that attains the worst-case bound on the length of a minimal spanning tree of k points. The factor of $(m^{-1} - \alpha)$ equals the side length of Q_i^α, and it reflects the scaling of $\rho_{\mathrm{MST}}(k)$ down to the smaller cube. Next, we let T be a minimal spanning tree of the set of $m^d k$ points $\bigcup_{i=1}^{m^d} S_i$, and we let T_i denote a minimal spanning tree of S_i. We will now develop a relationship between $L(T)$ and $L(\bigcup_{i=1}^{m^d} T_i)$ that moves us toward an inequality like (2.1(ii)) for ρ_{MST}.

First consider the forest that is obtained from T by deleting from T all the edges that have length as great as α. We let $\lambda(\alpha)$ denote the number of edges deleted from T, i.e., we set

$$\lambda(\alpha)=|\{e\in T: |e|\geqq \alpha\}|.$$

Since T was connected, the graph that remains following the deletion of $\lambda(\alpha)$ edges has at most $\lambda(\alpha)+1$ connected components. Moreover, each of these connected components is contained entirely in some subcell Q_i^α.

Next, if two or more connected components of T coexist in the same subcell Q_i^α, then we join them together to make a tree on the point set S_i. Since, within any given cell, we can rejoin any two components at a cost not exceeding $d^{1/2}m^{-1}$, the total cost of rejoining all the within-cell components is bounded by $d^{1/2}m^{-1}\lambda(\alpha)$.

So far we have constructed a spanning tree for each S_i, where $1\leqq i\leqq m^d$. Since the length of each of these trees must be at least as great as the length of the minimal spanning tree T_i of the point set S_i, we have the bound

$$\sum_{i=1}^{m^d} L(T_i)\leqq L(T)+d^{1/2}m^{-1}\lambda(\alpha).$$

But we know $L(T_i)=(m^{-1}-\alpha)\rho_{\mathrm{MST}}(k)$ and $L(T)\leqq \rho_{\mathrm{MST}}(m^dk)$, so we can rewrite this bound to provide

$$m^d(m^{-1}-\alpha)\rho_{\mathrm{MST}}(k)-d^{1/2}m^{-1}\lambda(\alpha)\leqq \rho_{\mathrm{MST}}(m^dk). \tag{3.1}$$

In order to extract an equality like (2.1(ii)) for ρ_{MST} from (3.1), we need some elementary facts about sets of points in $[0, 1]^d$ and their associated minimal spanning trees. We begin by recalling an easy pigeonhole argument, which says that from any set of n points in $[0, 1]^d$, one can always find a pair that are close together.

LEMMA 3.1. *There exists a constant c_2 such that for any $\{x_1, x_2, \cdots, x_n\}\subset[0, 1]^d$, where $n\geqq 2$, one has*

$$|x_i-x_j|\leqq c_2 n^{-1/d}$$

for some x_i and x_j, $1\leqq i<j\leqq n$.

Proof. Cover each x_i with a ball of radius r centered at x_i, and note that such a ball has volume $\omega_d r^d$, where ω_d is the volume of the unit d-ball. If all of the balls constructed in this way were non-intersecting, then each would cover at least $2^{-d-1}\omega_d r^d$ of $[0, 1]^d$, even if we generously assume that each of the balls were centered exactly in a corner of the hypercube. In total, the n balls would cover at least a volume of $2^{-d-1}\omega_d r^d n$, and since $[0, 1]^d$ has unit volume, we have $2^{-d-1}\omega_d r^d n\leqq 1$. The lemma is therefore established with $c_2=2^{(2d+1)/d}\omega_d^{-1/d}$.

One can easily improve the constant c_2, but this simply derived constant is sufficient for our purposes. It is now easy to give a bound on ρ_{MST} that shows the validity of the first hypothesis of Lemma 2.1.

LEMMA 3.2. *There exists a constant c_3 such that for all $n\geqq 1$, one has the bound*

$$\rho_{\mathrm{MST}}(n+1)\leqq \rho_{\mathrm{MST}}(n)+c_3 n^{-1/d}. \tag{3.2}$$

Proof. Let $S=\{x_1, x_2, \cdots, x_{n+1}\}$ denote a set of $n+1$ points in $[0, 1]^d$ for which the length of a minimal spanning tree is $\rho_{\mathrm{MST}}(n+1)$. By Lemma 3.1, there exist x_i and x_j in S such that $|x_i-x_j|\leqq c_2(n+1)^{-1/d}\leqq c_2 n^{-1/d}$. We form a minimal spanning tree T of $\{x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1}\}$ and then augment the tree by adding to it the edge

$\{x_i, x_j\}$. This construction provides a spanning tree of S at a cost of no more than $L(T)+c_2n^{-1/d}$. Therefore, we have

$$\rho_{\text{MST}}(n+1) \leqq L(T)+c_2n^{-1/d} \leqq \rho_{\text{MST}}(n)+c_2n^{-1/d},$$

which proves our lemma with $c_3=c_2$.

Naturally we can sum inequality (3.2) to provide a bound on $\rho_{\text{MST}}(n)$.

COROLLARY. *There is a constant c_4 such that for all $n \geqq 1$, one has the bound*

$$\rho_{\text{MST}}(n) \leqq c_4 n^{(d-1)/d}. \tag{3.3}$$

Here we note that $c_4 = 2c_3$ is a sufficient choice for the constant c_4.

The next lemma provides a tool for understanding how a minimal spanning tree changes as edges are added or deleted. While the result is reasonably intuitive and can be established by modification of Kruskal's algorithm (see, e.g., Aho, Hopcroft, and Ullman (1974)), the rigorous justification of the modified Kruskal algorithm does not seem to be as easy as the characterization-based proof used here.

LEMMA 3.3. *Let E be a subset of a minimal spanning tree of $S=\{x_1, x_2, \cdots, x_n\} \subset [0,1]^d$, and let S' be the set of points incident with the edges of E. Then, there exists a minimal spanning tree of S' that contains E.*

Proof. The graph corresponding to the set E consists of k connected components $(S_1, T_1), (S_2, T_2), \cdots, (S_k, T_k)$, where $1 \leqq k \leqq |E|$. We first show that for all $1 \leqq i \leqq k$, T_i is a minimal spanning tree of S_i. To see this, consider a minimal spanning tree T. If we form a forest of two trees by removing an edge from T, then it is trivial to note that each resulting tree is a minimal spanning tree of the respective set of points incident with it. Now let T be a minimal spanning tree of S, and recursively apply this fact by removing from the tree T all the edges of $T-E$. As each edge $e \in T-E$ is removed, the minimal spanning tree to which e belongs becomes two minimal spanning trees. After removing all the edges of $T-E$, the result is the edge set E, which is a forest of minimal spanning trees.

We first recall a well-known fundamental property of minimal spanning trees. If $\{(V_1, E_1), (V_2, E_2), \cdots, (V_k, E_k)\}$, where $k>1$, is a forest spanning the point set S, and $e=\{x_i, x_j\}$ is an edge of minimum length such that e has exactly one endpoint in V_1, then there exists a tree T^* spanning S and including $\bigcup_{i=1}^k E_i \cup \{e\}$ such that $L(T^*) = \min\{L(T)\colon T \text{ is a tree spanning } S \text{ and } \bigcup_{i=1}^k E_i \subset T\}$. We use this easily proved fact (see Aho, Hopcroft, and Ullman (1974), or Papadimitriou and Steiglitz (1982)) to construct from the edge set E a minimal spanning tree of S'.

Begin with the edges of E, which constitute a forest of minimal spanning trees, and iteratively add to T_1 an edge of minimal length over all those edges having exactly one endpoint in S_1. Merging components this way, we obtain a tree T that spans S'. Moreover, T is a minimum-cost tree over all trees that span S' and contain E. Hence, the only way we could lessen the cost of T would be lessen the cost of a tree T_i, where $1 \leqq i \leqq k$. But, since T_i is a minimal spanning tree, this is impossible, and we conclude that T is a minimal spanning tree of S'. Since T contains E, the proof is complete.

We now use Lemma 3.3 and the corollary to Lemma 3.2 to bound the total length of any k edges of a minimal spanning tree.

LEMMA 3.4. *There is a constant c_5 such that if E is any subset of the edges of a minimal spanning tree of $\{x_1, x_2, \cdots, x_n\} \subset [0,1]^d$, then*

$$L(E) \leqq c_5 |E|^{(d-1)/d}.$$

Proof. Let S be the set of endpoints of the edges of E and note $|S| \leqq 2|E|$. By Lemma 3.3, there exists a minimal spanning tree of S that contains E. By inequality

(3.3) we have therefore that

$$\sum_{e\in E} |e| \leqq c_4 |S|^{(d-1)/d},$$

so the lemma is proved with $c_5 = 2^{(d-1)/d} c_4$.

We require one more general inequality in order to bound $\lambda(\alpha)$ in our key relation (3.1). Formally, we let $\nu_{\mathrm{MST}}(x)$ denote the maximal value k such that there exists a minimal spanning tree of some $V_n = \{x_1, x_2, \cdots, x_n\} \subset [0, 1]^d$ with k edges greater than or equal to x in length.

LEMMA 3.5. *There is a constant c_6 such that for all $x > 0$, one has*

$$\nu_{\mathrm{MST}}(x) \leqq c_6 x^{-d}. \tag{3.4}$$

Proof. Let T be a minimal spanning tree of $\{x_1, x_2, \cdots, x_n\}$, and set $\phi_T(x) = |\{e \in T \colon |e| \geqq x\}|$. By Lemma 3.4 any set of $\phi_T(x)$ edges of T must have length bounded by $c_5 \phi_T(x)^{(d-1)/d}$, so

$$x\phi_T(x) \leqq \sum_{\substack{|e|\geqq x\\ e\in T}} |e| \leqq c_5 \phi_T(x)^{(d-1)/d}. \tag{3.5}$$

Clearing ϕ_T to the left side gives

$$\phi_T(x) \leqq c_5^d x^{-d},$$

and, since this bound holds for all minimal spanning trees T, the lemma is proved with $c_6 = c_5^d$.

Returning to our basic recurrence relation (3.1), we can write it in a form closer to the hypothesis of Lemma 2.1 as follows:

$$m^{d-1} \rho_{\mathrm{MST}}(k) - \{m^d \alpha \rho_{\mathrm{MST}}(k) + d^{1/2} m^{-1} \lambda(\alpha)\} \leqq \rho_{\mathrm{MST}}(m^d k). \tag{3.6}$$

By Lemma 3.2 and its corollary $\rho_{\mathrm{MST}}(k) \leqq c_4 k^{(d-1)/d}$, and by Lemma 3.5 $\lambda(\alpha) \leqq c_6 \alpha^{-d}$, so the bracketed expression of inequality (3.6) is majorized by

$$c_4 \alpha m^d k^{(d-1)/d} + c_6 d^{1/2} m^{-1} \alpha^{-d}.$$

This quantity is approximately minimized by choosing $\alpha = m^{-1} k^{(1-d)/d(d+1)}$, and making that choice proves that there is a constant c_7 such that the inequality

$$m^{d-1} \rho_{\mathrm{MST}}(k) - c_7 m^{d-1} k^{(d-1)/(d+1)} \leqq \rho_{\mathrm{MST}}(m^d k) \tag{3.7}$$

holds for all $m \geqq 1$ and $k \geqq 1$. This last inequality shows that the main hypothesis of Lemma 2.1 is valid with $r(k) = c_7 k^{(1-d)/d(d+1)}$. Since we already know that $\rho_{\mathrm{MST}}(n+1) \leqq \rho_{\mathrm{MST}}(n) + c_3 n^{-1/d}$, we have verified all of the hypotheses of Lemma 2.1. We have therefore proved that $\rho_{\mathrm{MST}}(n) \sim \beta_{\mathrm{MST}} n^{(d-1)/d}$ as $n \to \infty$ for all $d \geqq 2$.

To see that $\beta_{\mathrm{MST}} \geqq 1$, we just note that one can place n points in the unit d-cube in such a way that no two are closer together than $n^{-1/d}$. This proves that $\beta_{\mathrm{MST}} \geqq 1$, since any connected tree has $n-1$ edges.

4. The traveling salesman problem. Just as in the treatment of minimal spanning trees, the central task is to prove the validity of (2.1(ii)). For the traveling salesman problem the task actually turns out to be easier than it was for minimal spanning trees.

As before, we partition $[0, 1]^d$ into m^d cells Q_i of edge length m^{-1}. We then obtain a fattened grating H^α of width α, where $0 < \alpha < m^{-1}$, and define corresponding subcells Q_i^α with edge length $m^{-1} - \alpha$. Into each subcell Q_i^α we insert a set S_i of k points having an optimal traveling salesman tour with length $\rho_{\mathrm{TSP}}(k)(m^{-1} - \alpha)$, i.e., the set S_i attains the maximal length of any set of k points in a cube of edge length $m^{-1} - \alpha$.

Now, for each $1 \leqq i \leqq m^d$, we let T_i denote an optimal traveling salesman tour of S_i, and we further let T be an optimal traveling salesman tour of the $m^d k$ points of $\bigcup_{i=1}^{m^d} S_i$. We need to establish a relationship between the total lengths of the two sets of edges T and $\bigcup_{i=1}^{m^d} T_i$.

To build a heuristic tour T_i' through S_i, we start by taking the set T_i' to be E_i, the set of all of the edges of T that are completely contained in Q_i^α. If this set of edges forms a graph $G_i = (S_i, E_i)$ with k_i connected components, then there is a set C_i of at least k_i vertices that are in different components of G_i and have degree one or zero. The case of degree zero occurs exactly for those components consisting of a single vertex.

Since C_i has cardinality at least k_i, we can apply Lemma 3.1 to find a pair of vertices in C_i that are separated by a distance of at most $c_2 k_i^{-1/d}(m^{-1} - \alpha)$. We now add the edge determined by this pair of vertices to T_i'. Repeating this construction, we can add a total of $k_i - 1$ edges to E_i and obtain a path T_i' through all of the vertices in S_i. The ends of this path can now be joined by one final edge in order to complete the heuristic tour T_i'.

This process shows that the length of T_i' is bounded by

$$L(T_i') \leqq L(E_i) + c_2 \sum_{j=1}^{k_i} j^{-1/d} m^{-1}. \tag{4.1}$$

Now, since $\rho_{\mathrm{TSP}}(k)(m^{-1} - \alpha) \leqq L(T_i')$ and $\sum_{i=1}^{m^d} L(E_i) \leqq L(T) \leqq \rho_{\mathrm{TSP}}(m^d k)$, we can sum (4.1) over $1 \leqq i \leqq m^d$ and obtain

$$m^d \rho_{\mathrm{TSP}}(k)(m^{-1} - \alpha) \leqq \rho_{\mathrm{TSP}}(m^d k) + c_2 m^{-1} \sum_{i=1}^{m^d} \sum_{j=1}^{k_i} j^{-1/d}. \tag{4.2}$$

Next, let $\lambda(\alpha)$ denote the number of edges of T that intersect H^α. The number of connected components of $\bigcup_{i=1}^{m^d} G_i$ equals $\sum_{i=1}^{m^d} k_i \leqq \lambda(\alpha)$; so, estimating the inner sum of (4.2) by $\sum_{j=1}^{k_i} j^{-1/d} \leqq 1 + \int_1^{k_i} x^{-1/d}\,dx \leqq 2k_i^{(d-1)/d}$ and then applying Hölders inequality, we have

$$\begin{aligned} m^d \rho_{\mathrm{TSP}}(k)(m^{-1} - \alpha) &\leqq \rho_{\mathrm{TSP}}(m^d k) + 2c_2 m^{-1} \sum_{i=1}^{m^d} k_i^{(d-1)/d} \\ &\leqq \rho_{\mathrm{TSP}}(m^d k) + 2c_2 m^{-1} \left(\sum_{i=1}^{m^d} 1 \right)^{1/d} \left(\sum_{i=1}^{m^d} k_i \right)^{(d-1)/d} \\ &\leqq \rho_{\mathrm{TSP}}(m^d k) + 2c_2 \lambda(\alpha)^{(d-1)/d}. \end{aligned} \tag{4.3}$$

This inequality will now be put in the form needed to verify (2.1(ii)). The only real issue which remains is that of bounding $\lambda(\alpha)$, but some intermediate facts are required. First, we note that we can show

$$\rho_{\mathrm{TSP}}(n+1) \leqq \rho_{\mathrm{TSP}}(n) + c_8 n^{-1/d} \tag{4.4}$$

by taking $n+1$ points S such that $\rho_{\mathrm{TSP}}(n+1)$ is the length of the shortest tour through S and then using Lemma 3.1 to exhibit a heuristic tour through S with cost bounded by $\rho_{\mathrm{TSP}}(n) + 2c_2 n^{-1/d}$, so we have inequality (4.4) with $c_8 = 2c_2$. (For examples of this type of argument, where more attention is given to obtaining good values for the associated constants, one can consult Moran (1984). For an easier, but less quantitative, version one can consult Few (1955).)

One immediate consequence of (4.4) is that by summing over $1 \leqq i \leqq n$, we have

$$\rho_{\mathrm{TSP}}(n) \leqq c_9 n^{(d-1)/d}, \tag{4.5}$$

where $c_9 = 2c_8$.

Now, for an edge of T to intersect H^α, it must have endpoints in two different Q_i^α and therefore have length at least α. This gives the bound

$$\alpha\lambda(\alpha) \leqq \sum_{\substack{|e| \geqq \alpha \\ e \in T}} |e| \leqq \rho_{\mathrm{TSP}}(m^d k) \leqq c_9 m^{d-1} k^{(d-1)/d}. \tag{4.6}$$

When we apply (4.5) and (4.6) to (4.3) we have

$$m^{d-1}\rho_{\mathrm{TSP}}(k) \leqq \rho_{\mathrm{TSP}}(m^d k) + \alpha m^d c_9 k^{(d-1)/d} + c_{10}\alpha^{-(d-1)/d} m^{(d-1)^2/d} k^{(d-1)^2/d^2}, \tag{4.7}$$

where $c_{10} = 2c_2 c_9^{(d-1)/d}$. If we now choose $\alpha = m^{-1}k^{(1-d)/d(2d-1)} < m^{-1}$, we see that (4.7) simplifies to give

$$m^{d-1}\rho_{\mathrm{TSP}}(k) \leqq \rho_{\mathrm{TSP}}(m^d k) + c_{11} m^{d-1} k^{2(d-1)^2/d(2d-1)}, \tag{4.8}$$

for a constant c_{11}.

From inequality (4.8) we see that the main hypothesis of Lemma 2.1 is justified with $r(k) = c_{11} k^{(1-d)/d(2d-1)}$. Since (4.4) verifies the first hypothesis of Lemma 2.1, we have completed the proof that $\rho_{\mathrm{TSP}}(n) \sim \beta_{\mathrm{TSP}} n^{(d-1)/d}$ as $n \to \infty$. Naturally, since the minimal spanning tree problem is a relaxation of the traveling salesman problem, we have $\beta_{\mathrm{MST}} \leqq \beta_{\mathrm{TSP}}$.

5. Summary and concluding remarks. The two classical examples that were studied here follow a general pattern that can be used for other problems. One pursues the following recipe: (1) divide the unit d-cube into m^d subcells of equal size that are separated by a fattened grating; (2) fill the ith subcell with S_i, a set of k points on which the geometric object being analyzed attains its worst-case length in the subcell; (3) construct a graph G that is associated with the points $\bigcup_{i=1}^{m^d} S_i \subset [0, 1]^d$; (4) delete all edges of G that are long enough to span the fattened grating; (5) in each subcell, add edges to what remains following the deletion to form a heuristic graph G_i on S_i; (6) derive from this construction a recursion involving the length of a worst-case edge set; and (7) show that the recursion justifies (2.1(ii)) of Lemma 2.1. Of course, we must also show that the worst-case length satisfies (2.1(i)) of Lemma 2.1 to guarantee the result, although proving that the second recursion of Lemma 2.1 is satisfied is usually the task of greater difficulty.

This recipe would be unacceptably vague in the absence of explicit examples, but, be referring to the detailed treatment of the MST and TSP, the application of this technique to other problems should be reasonably straightforward.

The fact that the traveling salesman problem is computationally difficult and the minimal spanning tree problem is computationally easy serves to show that computational complexity is not at the heart of the technique used here. This intriguing circumstance provided one of our motivations for illustrating our technique with these particular problems. A second motivation came from the heuristic algorithms developed by Held and Karp (1970), (1971) which are driven by the observation that the minimal spanning tree problem is a relaxation of the traveling salesman problem.

Limit results like those given here seem to provoke two inevitable questions. The first question concerns the determination of the constants β_{MST} and β_{TSP} (for each $d \geqq 2$), and the second concerns the possibility of providing convergence rates more precise than $\rho(n) = \beta n^{(d-1)/d} + o(n^{(d-1)/d})$. The experience of trying to deal with the analogous questions under probabilistic assumptions leaves us with little hope for progress on these points. In particular, one should note that to sharpen the results of Moran (1984) to give the exact value of β_{TSP} would seem to require new geometric insights into the traveling salesman problem as well as improvements on the best available results on sphere packing. These steps would be major advances in their own right. Perhaps the problem of improving the error term in our limit theorem to something sharper than $o(n^{(d-1)/d})$ would be easier than determining β; but, still, one would have to develop a technique that would be completely different than that given here.

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