A Note on a Decreasing Property of t Confidence Intervals

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Abstract

Let $X_1, X_2, \ldots, X_n$ be iid $\mathcal{N}(\theta, \sigma^2)$ random variables, where both θ and σ^2 are unknown. A t-based confidence interval for θ can be constructed as follows. Partition $X_1, \ldots, X_n$ into m equal-sized groups (assuming n is a multiple of m), and denote the average of each group by $\overline{X}_1, \ldots, \overline{X}_m$, which are iid $\mathcal{N}(\theta, m\sigma^2/n)$. Using these m random variables, the t confidence interval at level $1-\alpha$ for θ takes the form

$$\left[\overline{X} - t_{m-1,1-\frac{\alpha}{2}} \frac{S_m}{\sqrt{m}}, \quad \overline{X} + t_{m-1,1-\frac{\alpha}{2}} \frac{S_m}{\sqrt{m}}\right],$$

where

$$\overline{X} = \frac{1}{m}\sum_{i=1}^{m} \overline{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_j \text{ and } S_m^2 = \frac{1}{m-1}\sum_{i=1}^{m}(\overline{X}_i - \overline{X})^2.$$

(Note that the case of $m = n$, this reduces to the conventional t confidence interval.) The expected length of the confidence interval is

$$\begin{aligned} \mathbb{E}\left[2t_{m-1,1-\frac{\alpha}{2}} \frac{S_m}{\sqrt{m}}\right] &= 2t_{m-1,1-\frac{\alpha}{2}} \frac{\sqrt{m\sigma^2/n}\,\mathbb{E}\,\chi_{m-1}/\sqrt{m-1}}{\sqrt{m}} \\ &= \frac{2\sigma}{\sqrt{n}} \cdot \frac{t_{m-1,1-\frac{\alpha}{2}}\Gamma(\frac{m}{2})}{\sqrt{m-1}\Gamma(\frac{m-1}{2})}. \end{aligned}$$

This note aims to prove that, with fixed $0 < \alpha < 1$, the expected length above is decreasing in m.

1 Theorem Statement

Theorem 1. *For any fixed* $0 < \alpha < 1$*, the expression*

$$\frac{t_{m-1,1-\frac{\alpha}{2}}\Gamma\left(\frac{m}{2}\right)}{\sqrt{m-1}\,\Gamma\left(\frac{m-1}{2}\right)}$$

as a function of m *is decreasing in* $m \geq 2$*.*

2 Proof

This theorem amounts to showing that

$$\frac{t_{m-1,1-\frac{\alpha}{2}}\Gamma\left(\frac{m}{2}\right)}{\sqrt{m-1}\,\Gamma\left(\frac{m-1}{2}\right)} > \frac{t_{m,1-\frac{\alpha}{2}}\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{m}\,\Gamma\left(\frac{m}{2}\right)}$$

for every $m \geq 2$. More generally, we aim to show in this appendix that

$$\frac{t_{d_1,1-\frac{\alpha}{2}}\Gamma\left(\frac{d_1+1}{2}\right)}{\sqrt{d_1}\,\Gamma\left(\frac{d_1}{2}\right)} > \frac{t_{d_2,1-\frac{\alpha}{2}}\Gamma\left(\frac{d_2+1}{2}\right)}{\sqrt{d_2}\,\Gamma\left(\frac{d_2}{2}\right)}$$

for all $d_2 > d_1 \geq 1$. This inequality can be rewritten as

$$\frac{t_{d_1,1-\frac{\alpha}{2}}}{t_{d_2,1-\frac{\alpha}{2}}} > \frac{\sqrt{d_1}\,\Gamma\left(\frac{d_1}{2}\right)\Gamma\left(\frac{d_2+1}{2}\right)}{\sqrt{d_2}\,\Gamma\left(\frac{d_2}{2}\right)\Gamma\left(\frac{d_1+1}{2}\right)}. \tag{2.1}$$

To establish (2.1) is the subject of the lemma below.

Lemma 2.1. *Let $1 \leq d_1 < d_2$ be two integers. Then,* (2.1) *holds for all* $0 < \alpha < 1$.

Proof of Lemma 2.1. Suppose on the contrary that

$$\frac{t_{d_1,1-\frac{\alpha_0}{2}}}{t_{d_2,1-\frac{\alpha_0}{2}}} \leq \frac{\sqrt{d_1}\Gamma\left(\frac{d_1}{2}\right)\Gamma\left(\frac{d_2+1}{2}\right)}{\sqrt{d_2}\Gamma\left(\frac{d_2}{2}\right)\Gamma\left(\frac{d_1+1}{2}\right)} \tag{2.2}$$

for some $\alpha_0 \in (0,1)$. Denote by $\lambda = \frac{t_{d_1,1-\frac{\alpha_0}{2}}}{t_{d_2,1-\frac{\alpha_0}{2}}}$. Note that λ must be larger than 1 since $d_2 > d_1$. Write $H_\lambda(x)$ for the difference between $\mathbb{P}(t_{d_1} \leq x)$ and $\mathbb{P}(t_{d_2} \leq x/\lambda)$ for $x > 0$. In particular, this function satisfies

$$\begin{aligned} H_\lambda(t_{d_1,1-\frac{\alpha_0}{2}}) &= \mathbb{P}(t_{d_1} \leq t_{d_1,1-\frac{\alpha_0}{2}}) - \mathbb{P}(t_{d_2} \leq t_{d_1,1-\frac{\alpha_0}{2}}/\lambda) \\ &= \mathbb{P}(t_{d_1} \leq t_{d_1,1-\frac{\alpha_0}{2}}) - \mathbb{P}(t_{d_2} \leq t_{d_2,1-\frac{\alpha_0}{2}}) \\ &= 1 - \frac{\alpha_0}{2} - \left(1 - \frac{\alpha_0}{2}\right) \\ &= 0. \end{aligned} \tag{2.3}$$

In general, denoting the density of t_d by p_d, $H_\lambda(x)$ takes the following form

$$\begin{aligned} H_\lambda(x) &= \mathbb{P}(t_{d_1} \leq x) - \mathbb{P}(t_{d_2} \leq x/\lambda) \\ &= \int_{-\infty}^{x} p_{d_1}(u)\mathrm{d}u - \int_{-\infty}^{x/\lambda} p_{d_2}(u)\mathrm{d}u \\ &= \int_0^{x} p_{d_1}(u)\mathrm{d}u - \int_0^{x/\lambda} p_{d_2}(u)\mathrm{d}u \\ &= \int_0^{x} p_{d_1}(u) - p_{d_2}(u/\lambda)/\lambda\mathrm{d}u. \end{aligned}$$

Let $h_\lambda(u)$ be the integrand $p_{d_1}(u) - p_{d_2}(u/\lambda)/\lambda$. If one can show that

$$H_\lambda(x) = \int_0^x h_\lambda(u)\mathrm{d}u < 0 \tag{2.4}$$

for all $x > 0$ given

$$\lambda \le \frac{\sqrt{d_1}\Gamma\left(\frac{d_1}{2}\right)\Gamma\left(\frac{d_2+1}{2}\right)}{\sqrt{d_2}\Gamma\left(\frac{d_2}{2}\right)\Gamma\left(\frac{d_1+1}{2}\right)}$$

that follows from the assumption (2.2), we get a contradiction to (2.3). Consequently, (2.2) cannot be satisfied.

Below, Lemma 2.2 affirms (2.4) as the last step to prove the present lemma, thus, concluding the proof of Theorem 1.

□

Lemma 2.2. *If*

$$1 < \lambda \le \frac{\sqrt{d_1}\Gamma\left(\frac{d_1}{2}\right)\Gamma\left(\frac{d_2+1}{2}\right)}{\sqrt{d_2}\Gamma\left(\frac{d_2}{2}\right)\Gamma\left(\frac{d_1+1}{2}\right)},$$

then $H_\lambda(x) < 0$ *for all* $x > 0$.

Proof of Lemma 2.2. Note that $H_\lambda(x)$ results from integrating $h_\lambda(x)$. The sign of $h_\lambda(x)$ depends on whether the ratio

$$\begin{aligned}\frac{\frac{\Gamma\left(\frac{d_1+1}{2}\right)}{\sqrt{\pi d_1}\Gamma\left(\frac{d_1}{2}\right)}\left(1+\frac{x^2}{d_1}\right)^{-\frac{d_1+1}{2}}}{\frac{\Gamma\left(\frac{d_2+1}{2}\right)}{\lambda\sqrt{\pi d_2}\Gamma\left(\frac{d_2}{2}\right)}\left(1+\frac{x^2}{\lambda^2 d_2}\right)^{-\frac{d_2+1}{2}}} &= \frac{\lambda\sqrt{d_2}\Gamma\left(\frac{d_2}{2}\right)\Gamma\left(\frac{d_1+1}{2}\right)}{\sqrt{d_1}\Gamma\left(\frac{d_1}{2}\right)\Gamma\left(\frac{d_2+1}{2}\right)}\cdot\frac{\left(1+\frac{x^2}{\lambda^2 d_2}\right)^{\frac{d_2+1}{2}}}{\left(1+\frac{x^2}{d_1}\right)^{\frac{d_1+1}{2}}}\\ &\equiv C_\lambda\frac{\left(1+\frac{x^2}{\lambda^2 d_2}\right)^{\frac{d_2+1}{2}}}{\left(1+\frac{x^2}{d_1}\right)^{\frac{d_1+1}{2}}}\end{aligned}$$

exceeds 1 or not. Above, we use the fact that the density of the t distribution with d degrees of freedom reads

$$p_d(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi d}\Gamma\left(\frac{d}{2}\right)}\left(1+\frac{x^2}{d}\right)^{-\frac{d+1}{2}}$$

for $x > 0$. Denote by this ratio $r_\lambda(x)$. It is clear that $r_\lambda(0) = C_\lambda \le 1$, implying that $h_\lambda(0) \le 0$. In addition, as $x \to \infty$, the ratio $r_\lambda(x) \to \infty$ as well, and this reveals that $h_\lambda(x) > 0$ for sufficiently large x.

To get a closer look, note that

$$\begin{aligned}\frac{\mathrm{d}\log r_\lambda(x)}{\mathrm{d}x} &= \frac{d_2+1}{2}\frac{2x}{\lambda^2 d_2 + x^2} - \frac{d_1+1}{2}\frac{2x}{d_1+x^2}\\ &= \frac{(d_2-d_1)x^3 - \left[(\lambda^2-1)d_1 d_2 + \lambda^2 d_2 - d_1\right]x}{(\lambda^2 d_2 + x^2)(d_1+x^2)}.\end{aligned}$$

The fact that $\lambda > 1$ and $d_2 > d_1$ ensures that $(\lambda^2 - 1)d_1 d_2 + \lambda^2 d_2 - d_1 > 0$. Hence, we get

$$\frac{\mathrm{d}\log r_\lambda(x)}{\mathrm{d}x} < 0, \text{ if } 0 < x < \sqrt{\frac{(\lambda^2 - 1)d_1 d_2 + \lambda^2 d_2 - d_1}{d_2 - d_1}}$$
$$\frac{\mathrm{d}\log r_\lambda(x)}{\mathrm{d}x} > 0, \text{ if } x > \sqrt{\frac{(\lambda^2 - 1)d_1 d_2 + \lambda^2 d_2 - d_1}{d_2 - d_1}}.$$

Thus, $r_\lambda(x)$, starting from $r_\lambda(0) \leq 1$, stays below 1 for $0 < x < x_0$ and then stays above 1 for $x_0 < x < \infty$, where $x_0 > 0$ is some number determined by d_1, d_2 and λ. Put differently, the above discussion demonstrates that

$$\begin{aligned} &h_\lambda(x) < 0, \text{ for } 0 < x < x_0 \\ &h_\lambda(x) > 0, \text{ for } x_0 < x < \infty. \end{aligned} \tag{2.5}$$

Having established (2.5), it is a stone's throw away to prove the lemma. If $x < x_0$, then (2.5) readily gives

$$H_\lambda(x) = \int_0^x h_\lambda(u)\mathrm{d}u < 0.$$

In the case where $x > x_0$, (2.5) together with the fact that $H_\lambda(\infty) = 0$ gives

$$H_\lambda(x) = \int_0^x h_\lambda(u)\mathrm{d}u = \int_0^\infty h_\lambda(u)\mathrm{d}u - \int_x^\infty h_\lambda(u)\mathrm{d}u = -\int_x^\infty h_\lambda(u)\mathrm{d}u < 0,$$

as desired.

□